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Cambridge International
AS & A Level Mathematics:

Pure Mathematics 2 & 3

Worked Solutions Manual

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Contents

How to use this resource

Welcome to your Cambridge Elevate Worked Solutions Manual

This resource contains worked solutions to the questions in the Cambridge International AS & A Level Mathematics: Pure Mathematics 2 & 3 Coursebook. This includes questions from the chapter exercises, end-of-chapter review exercises, cross-topic review exercises and practice exam-style papers.

This resource covers both Pure Mathematics 2 and Pure Mathematics 3. Exercise 5E is only covered in Pure Mathematics 2 and this exercise is marked with the icon P2. Chapters 7–11 are only covered in Pure Mathematics 3 and these are marked with the icon P3. The icons appear in the Contents list and in the relevant sections of the resource.

Most of the chapter exercises include questions to help develop your fluency in solving a particular type of problem: these questions have multiple sub-parts that require you to practise the same procedures. For these fluency questions, we have included worked solutions for only some of the sub-parts. The solutions that are provided can be used for guidance about the steps required for the other question sub-parts. The aim of this is to encourage you to develop as a confident, independent thinker.

Each solution shows you step-by-step how to solve the question. You will be aware that often questions can be solved by multiple different methods. In this book, we provide a single method for each solution. Do not be disheartened if the working in a solution does not match your own working; you may not be wrong but simply using a different method. It is good practice to challenge yourself to think about the methods you are using and whether there may be alternative methods.

Additional guidance is included in **Commentary** boxes throughout the book. These boxes often clarify common misconceptions or areas of difficulty.

> The original equation in part **f** requires x to be positive, because $3x$ is equal to a modulus function. This is why the false solution $x = -\frac{7}{5}$ was not a correct solution.

Some questions in the coursebook go beyond the syllabus. We have indicated these solutions with a red line to the left of the text:

EXERCISE 2H

E

1 **e** $a = e^{x^2+by}$

Taking natural logs of both sides:

$$\ln a = x^2 + by$$
$$by = -x^2 + \ln a$$
$$y = -\frac{1}{b}x^2 + \frac{1}{b}\ln a$$
$$Y = y$$
$$m = -\frac{1}{b}$$
$$X = x^2$$
$$c = \frac{1}{b}\ln a$$

> Note that this answer appears to be different to the answer given in the coursebook. This is because in this worked solution it has been chosen to keep Y and y on the same axis, as has been done with X and x. The coursebook version uses $Y = x^2$, effectively swapping axes.

To navigate within the resource, select the relevant section from the Contents page and you will be taken to the page.

Please note that all worked solutions available for the Pure Mathematics 2 and 3 course can be found within this digital resource. Select material can also be found within the print resource.

All worked solutions in this resource have been written by the author. In examinations, the way marks are awarded may be different.

Chapter 1
Algebra

EXERCISE 1A

1 **a** $4x - 3 = 7$ or $4x - 3 = -7$

$$4x = 10 \qquad 4x = -4$$

$$x = \frac{5}{2} \qquad x = -1$$

Check:

$$\left|4 \times \frac{5}{2} - 3\right| = |7| = 7 \checkmark \qquad |4 \times (-1) - 3| = |-7| = 7 \checkmark$$

Solutions are:

$$x = \frac{5}{2} \quad \text{or} \quad x = -1$$

c $\frac{3x-2}{5} = 4$ or $\frac{3x-2}{5} = -4$

$$3x - 2 = 20 \qquad 3x - 2 = -20$$

$$3x = 22 \qquad 3x = -18$$

$$x = \frac{22}{3} \qquad x = -6$$

Check:

$$\left|\frac{3 \times \frac{22}{3} - 2}{5}\right| = |4| = 4 \checkmark \qquad \left|\frac{3 \times (-6) - 2}{5}\right| = \left|-\frac{20}{5}\right| = |-4| = 4 \checkmark$$

Solutions are:

$$x = \frac{22}{3} \quad \text{or} \quad x = -6$$

e $\frac{x+2}{3} - \frac{2x}{5} = 2$ or $\frac{x+2}{3} - \frac{2x}{5} = -2$

$$5(x+2) - 3(2x) = 30 \qquad 5(x+2) - 3(2x) = -30$$

$$5x + 10 - 6x = 30 \qquad 5x + 10 - 6x = -30$$

$$x = -20 \qquad x = 40$$

Check:

$$\left|\frac{-20+2}{3} - \frac{2 \times (-20)}{5}\right| = |-6 + 8| = |2| = 2 \checkmark$$

$$\left|\frac{40+2}{3} - \frac{2 \times 40}{5}\right| = |14 - 16| = |-2| = 2 \checkmark$$

Solutions are:

$$x = -20 \quad \text{or} \quad x = 40$$

f $2x + 7 = 3x$ or $2x + 7 = -3x$

$$x = 7 \qquad 5x = -7$$

$$x = -\frac{7}{5}$$

Check:

$$|2 \times 7 + 7| = |21| = 21, 3 \times 7 = 21 \checkmark$$

$$\left|2\times\left(-\frac{7}{5}\right)+7\right|=\left|-\frac{14}{5}+\frac{35}{5}\right|=\left|\frac{21}{5}\right|=\frac{21}{5}, 3\times\left(-\frac{7}{5}\right)=-\frac{21}{5}\neq\frac{21}{5}\ ✗$$

The only solution is $x=7$.

The original equation in part **f** requires x to be positive, because $3x$ is equal to a modulus function. This is why the false solution $x=-\frac{7}{5}$ was not a correct solution.

2 a

In the worked solution to part **a** two methods are shown. Unless instructed otherwise, you can choose the method that you are most confident with.

Method 1

$$\left|\frac{2x+1}{x-2}\right|=5$$

$$\frac{2x+1}{x-2}=5 \qquad \text{or} \qquad \frac{2x+1}{x-2}=-5$$

$$2x+1=5(x-2) \qquad 2x+1=-5(x-2)$$

$$2x+1=5x-10 \qquad 2x+1=-5x+10$$

$$3x=11 \qquad 7x=9$$

$$x=\frac{11}{3} \qquad x=\frac{9}{7}$$

Check:

$$\left|\frac{2\times\frac{9}{7}+1}{\frac{9}{7}-2}\right|=\left|\frac{\left(\frac{25}{7}\right)}{\left(-\frac{5}{7}\right)}\right|=\left|-\frac{25}{7}\times\frac{7}{5}\right|=|-5|=5\ ✓$$

$$\left|\frac{2\times\frac{11}{3}+1}{\frac{11}{3}-2}\right|=\left|\frac{\left(\frac{25}{3}\right)}{\left(\frac{5}{3}\right)}\right|=\left|\frac{25}{3}\times\frac{3}{5}\right|=|5|=5\ ✓$$

Solutions are:

$$x=\frac{9}{7}\text{ or }x=\frac{11}{3}$$

Method 2

$$\left|\frac{2x+1}{x-2}\right|=5\Rightarrow|2x+1|=5|x-2|$$

$$(2x+1)^2=25(x-2)^2$$

$$4x^2+4x+1=25x^2-100x+100$$

$$21x^2-104x+99=0$$

$$(7x-9)(3x-11)=0$$

$$x=\frac{9}{7}\text{ or }x=\frac{11}{3}$$

Solutions are:

$$x=\frac{9}{7}\text{ or }x=\frac{11}{3}$$

c

$$\left|2-\frac{x+2}{x-3}\right|=5$$

$$\left|\frac{2x-6-x-2}{x-3}\right|=5$$

$$|x-8|=5|x-3|$$

$$(x-8)^2=25(x-3)^2$$

$$x^2-16x+64=25x^2-150x+225$$

$$24x^2-134x+161=0$$

$$(4x-7)(6x-23)=0$$

$$x=\frac{7}{4}\text{ or }x=\frac{23}{6}$$

Check:

$$\left|2-\frac{\frac{7}{4}+2}{\frac{7}{4}-3}\right|=\left|2-\frac{\left(\frac{15}{4}\right)}{\left(-\frac{5}{4}\right)}\right|=\left|2+\frac{15}{4}\times\frac{4}{5}\right|=|2+3|=5\ \checkmark$$

$$\left|2-\frac{\frac{23}{6}+2}{\frac{23}{6}-3}\right|=\left|2-\frac{\left(\frac{35}{6}\right)}{\left(\frac{5}{6}\right)}\right|=\left|2-\frac{35}{6}\times\frac{6}{5}\right|=|2-7|=|-5|=5\ \checkmark$$

Solutions are:

$x=\frac{7}{4}$ or $x=\frac{23}{6}$

In part **c** the use of the common denominator allows you to multiply through and solve, as in part **a**.

e $x+|x+4|=8$

$|x+4|=8-x$

$x+4=8-x$ or $x+4=x-8$

$2x=4$ No solutions.

$x=2$

Check:

$2+|2+4|=2+6=8\ \checkmark$

The only solution is:

$x=2$

If you arrive at an equation that doesn't make sense then this will mean that there are no solutions, unless you have made a mistake in your working. You should check everything carefully if this happens.

3 **a**

In the worked solution to part **a** two methods are shown. Unless instructed otherwise, you can choose the method that you are most confident with.

Method 1

$|2x+1|=|x|$

$2x+1=x$ or $2x+1=-x$

$x=-1$ $3x=-1$

$x=-\frac{1}{3}$

Check:

$$\left|2\left(-\frac{1}{3}\right)+1\right|=\left|-\frac{2}{3}+1\right|=\left|\frac{1}{3}\right|=\frac{1}{3},\left|-\frac{1}{3}\right|=\frac{1}{3}\ \checkmark$$

$|2(-1)+1|=|-2+1|=|-1|=1, |-1|=1\ \checkmark$

Solutions are:

$x=-\frac{1}{3}$ or $x=-1$

Method 2

$$\begin{aligned}|2x+1|&=|x|\\(2x+1)^2&=x^2\\4x^2+4x+1&=x^2\\3x^2+4x+1&=0\\(3x+1)(x+1)&=0\end{aligned}$$

$x=-\frac{1}{3}$ or $x=-1$

Solutions are:

$x = -\frac{1}{3}$ or $x = -1$

d $|3x+5| = |1+2x|$

$3x + 5 = 1 + 2x$ or $3x + 5 = -(1+2x)$

$x = -4$ $\quad$ $3x + 5 = -1 - 2x$

$5x = -6$

$x = -\frac{6}{5}$

Check:

$\left|3\left(-\frac{6}{5}\right)+5\right| = \left|-\frac{18}{5}+5\right| = \left|\frac{7}{5}\right| = \frac{7}{5}, \left|1+2\left(-\frac{6}{5}\right)\right| = \left|1-\frac{12}{5}\right| = \left|-\frac{7}{5}\right| = \frac{7}{5}$ ✓

$|3(-4)+5| = |-12+5| = |-7| = 7, |1+2(-4)| = |1-8| = |-7| = 7$ ✓

Solutions are:

$x = -\frac{6}{5}$ or $x = -4$

f $3\,|2x-1| = \left|\frac{1}{2}x - 3\right|$

Multiplying both sides by 2:

$$\begin{aligned} 6\,|2x-1| &= |x-6| \\ 36(2x-1)^2 &= (x-6)^2 \\ 36(4x^2-4x+1) &= x^2-12x+36 \\ 144x^2-144x+36 &= x^2-12x+36 \\ 143x^2-132x &= 0 \\ x(143x-132) &= 0 \end{aligned}$$

$x = 0$ or $x = \frac{132}{143} = \frac{12}{13}$

Check:

$3\,|2(0)-1| = 3\,|-1| = 3 \times 1 = 3, \left|\frac{1}{2}(0)-3\right| = |-3| = 3$ ✓

$3\left|2\left(\frac{12}{13}\right)-1\right| = 3\left|\frac{11}{13}\right| = \frac{33}{13}, \left|\frac{1}{2}\left(\frac{12}{13}\right)-3\right| = \left|-\frac{33}{13}\right| = \frac{33}{13}$ ✓

Solutions are:

$x = 0$ or $x = \frac{132}{143} = \frac{12}{13}$

Always clear fractions where possible when solving equations.

4 **a** $x^2 - 2 = 7$ or $x^2 - 2 = -7$

$x^2 = 9$ $\quad$ $x^2 = -5$

$x = \pm 3$ $\quad$ No real solutions.

Check:

$|3^2-2| = |9-2| = 7$ ✓, $|(-3)^2-2| = |9-2| = 7$ ✓

Solutions are:

$x = \pm 3$

f $|x^2-7x+6| = 6-x$

$x^2 - 7x + 6 = 6 - x$ or $x^2 - 7x + 6 = x - 6$

$x^2 - 6x = 0$ $\quad$ $x^2 - 8x + 12 = 0$

$x(x-6) = 0$ $\quad$ $(x-6)(x-2) = 0$

$x = 0$ or $x = 6$ $\quad$ $x = 6$ or $x = 2$

Check:

$|0^2 - 7(0) + 6| = |6| = 6, 6 - 0 = 6$ ✓

$|6^2 - 7(6) + 6| = |0| = 0, 6 - 6 = 0$ ✓

$|2^2 - 7(2) + 6| = |-4| = 4, 6 - 2 = 4$ ✓

Solutions are:

$x = 0$ or $x = 6$ or $x = 2$

5 **a** $x + 2y = 8 \Rightarrow x = 8 - 2y$ .... [1]

$|x + 2| = 6 - y$ .... [2]

$x + 2 = 6 - y$ or $x + 2 = y - 6$

Substituting [1] into these separately:

$8 - 2y + 2 = 6 - y$, $y = 4$ $\qquad$ $8 - 2y + 2 = y - 6$, $y = \frac{16}{3}$

Substituting the values of y back into [1]:

$x = 8 - 2(4) = 0$ $\qquad$ $x = 8 - 2\left(\frac{16}{3}\right) = 8 - \frac{32}{3} = \frac{24}{3} - \frac{32}{3} = -\frac{8}{3}$

Solutions are:

$x = 0$ and $y = 4$ or $x = -\frac{8}{3}$ and $y = \frac{16}{3}$

b $3x + y = 0 \Rightarrow y = -3x$ .... [1]

$y = |2x^2 - 5|$

$2x^2 - 5 = y$ or $2x^2 - 5 = -y$

Substituting [1] into these separately:

$$\begin{aligned} 2x^2 - 5 &= -3x \\ 2x^2 + 3x - 5 &= 0 \\ (2x + 5)(x - 1) &= 0 \\ x = -\frac{5}{2} &\text{ or } x = 1 \end{aligned} \qquad \begin{aligned} 2x^2 - 5 &= 3x \\ 2x^2 - 3x - 5 &= 0 \\ (2x - 5)(x + 1) &= 0 \\ x = \frac{5}{2} &\text{ or } x = -1 \end{aligned}$$

Substituting these values back into [1] and remembering that $y \geqslant 0$ because it is equal to the modulus:

$y = \frac{15}{2}$ or $y = -3$(not a solution) $\qquad$ $y = -\frac{15}{2}$(not a solution) or $y = 3$

Solutions are:

$x = -\frac{5}{2}$ and $y = \frac{15}{2}$

or $x = -1$ and $y = 3$

6

$$\begin{aligned} 5|x - 1|^2 &= 2 - 9|x - 1| \\ 5(x - 1)^2 &= 2 - 9|x - 1| \\ 9|x - 1| &= 2 - 5(x - 1)^2 \end{aligned}$$

$$\begin{aligned} 9x - 9 &= 2 - 5(x - 1)^2 \\ 9x - 9 &= 2 - 5(x^2 - 2x + 1) \\ 5x^2 - x - 6 &= 0 \\ (5x - 6)(x + 1) &= 0 \\ x &= \frac{6}{5} \text{ or } -1 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 9x - 9 &= 5(x - 1)^2 - 2 \\ 9x - 9 &= 5(x^2 - 2x + 1) - 2 \\ 5x^2 - 19x + 12 &= 0 \\ (5x - 4)(x - 3) &= 0 \\ x &= \frac{4}{5} \text{ or } 3 \end{aligned}$$

Substitution shows that neither -1 nor 3 are legitimate solutions.

$x = \frac{6}{5}$ or $\frac{4}{5}$

Check:

$5\left|\frac{6}{5}-1\right|^2=5\left|\frac{1}{5}\right|^2=\frac{5}{25}=\frac{1}{5}, 2-9\left|\frac{6}{5}-1\right|=2-9\left|\frac{1}{5}\right|=2-\frac{9}{5}=\frac{1}{5}$ ✓

$5\left|\frac{4}{5}-1\right|^2=5\left|-\frac{1}{5}\right|^2=\frac{5}{25}=\frac{1}{5}, 2-9\left|\frac{4}{5}-1\right|=2-9\left|-\frac{1}{5}\right|=2-\frac{9}{5}=\frac{1}{5}$ ✓

Solutions are:

$x=\frac{6}{5}$ or $\frac{4}{5}$

Always check that your solutions work in the original equation. Squaring methods can sometimes generate false solutions that are not actually solutions at all.

7 **a** $x^2-5|x|+6=0$

$x^2-5x+6=0$ or $x^2+5x+6=0$

$(x-2)(x-3)=0$ $\quad$ $(x+2)(x+3)=0$

$x=2$ or $x=3$ $\quad$ $x=-2$ or $x=-3$

Check:

$2^2-5|2|+6=4-10+6=0$ ✓

$3^2-5|3|+6=9-15+6=0$ ✓

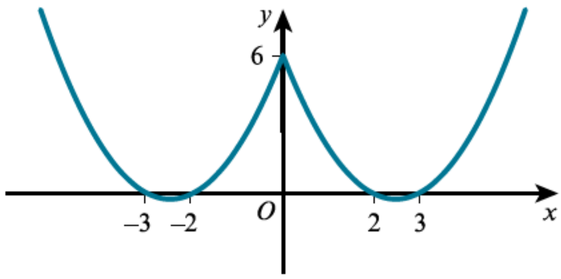
$(-2)^2-5|-2|+6=4-10+6=0$ ✓

$(-3)^2-5|-3|+6=9-15+6=0$ ✓

Solutions are:

$x=2$ or $x=3$ or $x=-2$ or $x=-3$

b



c y-axis or $x=0$

8 $|2x+1|+|2x-1|=3$

$|2x+1|=3-|2x-1|$

$2x+1=3-|2x-1|$ .... [1] $\quad$ or $\quad$ $2x+1=|2x-1|-3$ .... [2]

Considering [1] first:

$2x+1=3-|2x-1|$

$|2x-1|=3-2x-1$

$|2x-1|=2-2x$

$2x-1=2-2x$ $\quad$ or $\quad$ $2x-1=2x-2$

$4x=3$ $\quad$ No solution.

$x=\frac{3}{4}$

Considering [2] next:

$2x+1=|2x-1|-3$

$|2x-1|=2x+4$ $\quad$ or $\quad$ $2x-1=-2x-4$

$2x-1=2x+4$ $\quad$ $4x=-3$

No solutions. $\quad$ $x=-\frac{3}{4}$

Check:

$\left|2\left(\frac{3}{4}\right)+1\right|+\left|2\left(\frac{3}{4}\right)-1\right|=\left|\frac{10}{4}\right|+\left|\frac{2}{4}\right|=\frac{10}{4}+\frac{2}{4}=\frac{12}{4}=3$ ✓

$$\left|2\left(-\frac{3}{4}\right)+1\right|+\left|2\left(-\frac{3}{4}\right)-1\right|=\left|-\frac{2}{4}\right|+\left|-\frac{10}{4}\right|=\frac{2}{4}+\frac{10}{4}=\frac{12}{4}=3\ \checkmark$$

Solutions are:

$$x=\frac{3}{4}\text{ or }x=-\frac{3}{4}$$

9 $|3x-2y-11|=-2\sqrt{31-8x+5y}$

Given that the square root is always positive, the right-hand side is negative, unless it is zero. Given also that the modulus is positive or zero, this equation is only possible if both sides are zero at the same time.

$$3x-2y=11 \Rightarrow 15x-10y=55 \text{ .... [1]}$$
$$8x-5y=31 \Rightarrow 16x-10y=62 \text{ .... [2]}$$
$$[2]-[1]: \quad x=7$$

Substituting $x=7$ into $3x-2y=11$:

$$21-2y=11$$
$$2y=10$$
$$y=5$$

Check:

$$|3\times7-2\times5-11|=|21-10-11|=0$$
$$-2\sqrt{31-8\times7+5\times5}=-2\sqrt{31-56+25}=0\ \checkmark$$

The solution is:

$x=7$ and $y=5$

The square root symbol **always** means the positive square root. A common error, here, is to apply the '$\pm$' symbol, but that is only used when 'undoing' squares. For example, $x^2=9$ gives $x=\pm3$, whereas the square root of 9 is just 3.

EXERCISE 1B

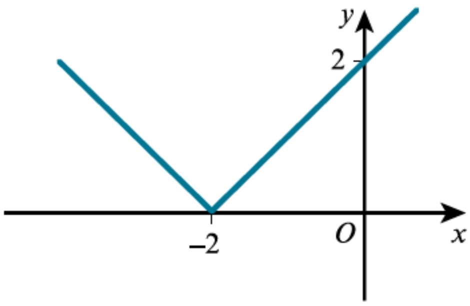
1 **a** $y = |x + 2|$

$x = 0 \Rightarrow y = |2| = 2$

$y = 0 \Rightarrow \quad x + 2 = 0$

$x = -2$

$$y = \begin{cases} x + 2 & x > -2 \\ -(x + 2) & x \leqslant -2 \end{cases}$$



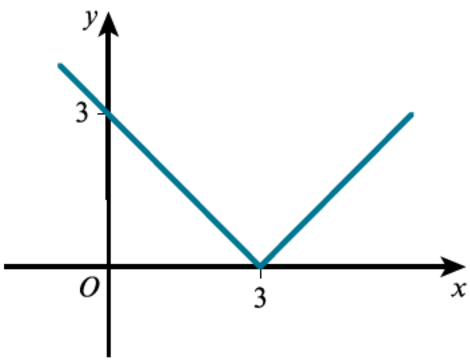
b $y = |3 - x|$

$x = 0 \Rightarrow y = |3| = 3$

$y = 0 \Rightarrow 3 - x = 0$

$x = 3$

$$y = \begin{cases} 3 - x & x < 3 \\ x - 3 & x \geqslant 3 \end{cases}$$



c $y = \left|5 - \frac{1}{2}x\right|$

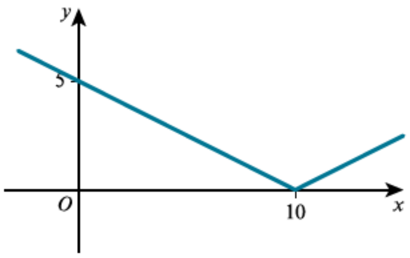
$x = 0 \Rightarrow y = |5| = 5$

$y = 0 \Rightarrow 5 - \frac{1}{2}x = 0$

$\frac{1}{2}x = 5$

$x = 10$

$$y = \begin{cases} 5 - \frac{1}{2}x & x < 10 \\ \frac{1}{2}x - 5 & x \geqslant 10 \end{cases}$$



In these solutions for Question 1, the point at which the graph crosses the x-axis has been found by seeing where $y = 0$. The point at which the graph crosses the y-axis has been found by seeing where $x = 0$.

2 **a** $x = 1 \Rightarrow\ y = |1 - 3| + 2 = 2 + 2 = 4$

$x = 3 \Rightarrow\ y = |3 - 3| + 2 = 2$

$x = 4 \Rightarrow\ y = |4 - 3| + 2 = 3$

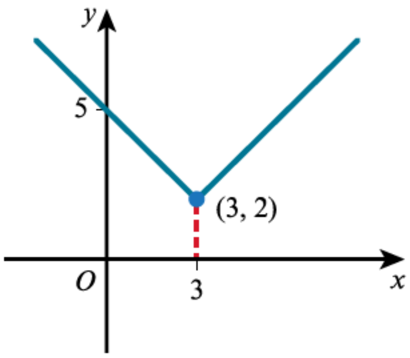
$x = 5 \Rightarrow\ y = |5 - 3| + 2 = 4$

$x = 6 \Rightarrow\ y = |6 - 3| + 2 = 5$

So the completed table is:

x	0	1	2	3	4	5	6
y	5	4	3	2	3	4	5

b



c $y = |x|$

$\downarrow$ translation $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$y = |x - 3|$

$\downarrow$ translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$y = |x - 3| + 2$

So the overall transformation is a translation $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

3

Sometimes it is easy to see what combination of transformations has taken place, but more complicated cases need to be built step by step. These worked solutions show one possible way of doing this.

a $y = |x|$

$\downarrow$ translation $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$y = |x + 1|$

$\downarrow$ translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$y = |x + 1| + 2$

So the overall transformation is a translation $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

e $y = |x|$

$\downarrow$ translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$y = |x + 2|$

$\downarrow$ reflection in $y = 0$

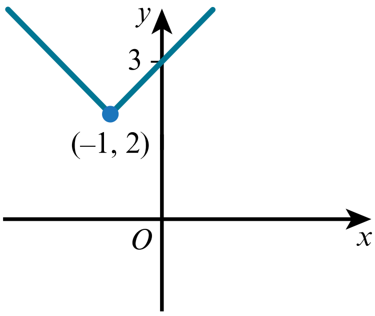
$y = -|x + 2|$

$\downarrow$ translation $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

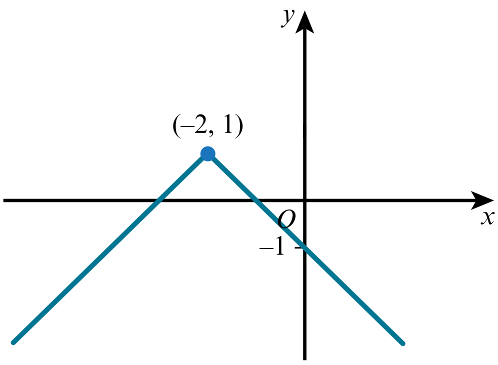
$y = 1 - |x + 2|$

So the overall sequence of transformations is a translation with vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ followed by a reflection in the line $y = 0$ followed by a translation with vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

4 **a**



e



5 $f(x) = |5 - 2x| + 3$

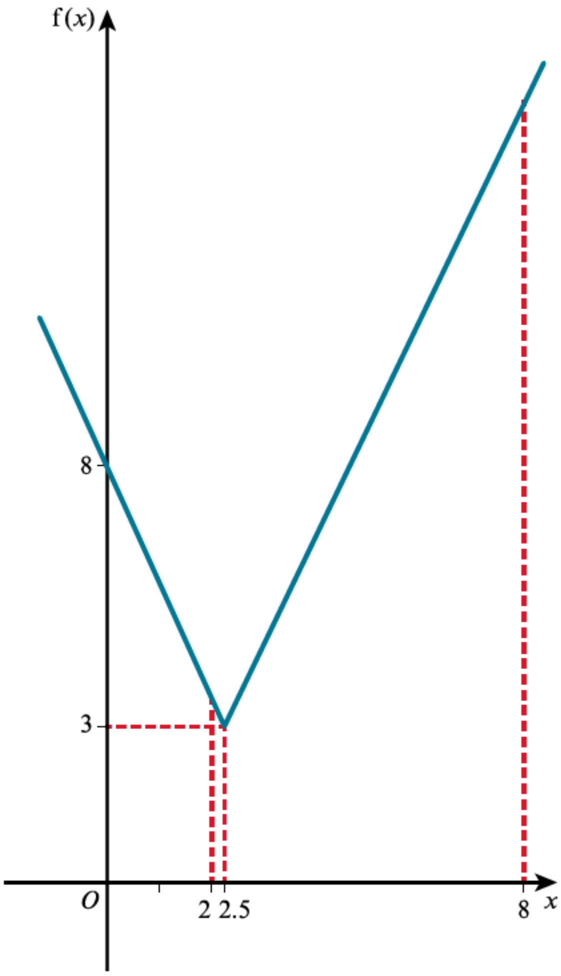
The minimum value is when $5 - 2x = 0$ because a modulus can't be negative.

So the minimum is when $x = 2.5$.

$f(0) = 5 + 3 = 8$

So the graph crosses the y-axis at 8.

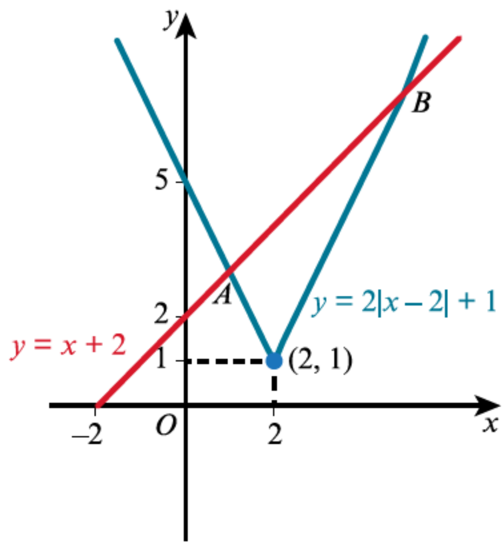
Draw the graph:



$f(2) = 1 + 3 = 4$
$f(8) = 11 + 3 = 14$
so $3 \leqslant f(x) \leqslant 14$

Remember that the range of a function is the set of all possible output values. Given that $f(x)$ is the notation used for the output, given an input x, you need to state the range using $f(x)$ and not just x.

6 **a, b**



c The solutions are the x-coordinates of the intersections of the graph of $y = 2|x - 2| + 1$ and the graph of $y = x + 2$. The graphs intersect at the points A and B.

At point A: At point B:

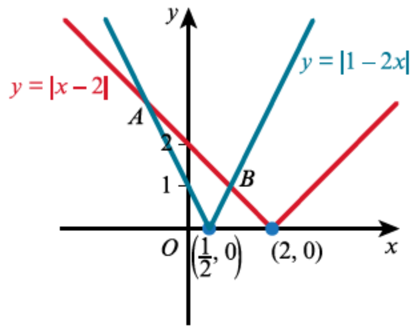
$x + 2 = -2(x - 2) + 1$
$x + 2 = -2x + 4 + 1$
$3x = 3$
$x = 1$

$x + 2 = 2(x - 2) + 1$
$x + 2 = 2x - 4 + 1$
$x = 5$

Solutions are: $x = 1$ or $x = 5$

> Take care to only change the sign of the term contained within the modulus, as opposed to the whole function.

7 **a, b**



c The solutions are the x-coordinates of the intersections of the graph of $y = |x - 2|$ and the graph of $y = |1 - 2x|$. The graphs intersect at the points A and B.

At point A:

$1 - 2x = -(x - 2)$
$1 - 2x = -x + 2$
$x = -1$

At point B:

$-(1 - 2x) = -(x - 2)$
$-1 + 2x = -x + 2$
$3x = 3$
$x = 1$

Solutions are: $x = 1$ or $x = -1$

8 **a** $x \geqslant 1$, so $x - 1 \geqslant 0$ and $x + 1 \geqslant 0$

$y = x + 1 + x - 1$
$= 2x$

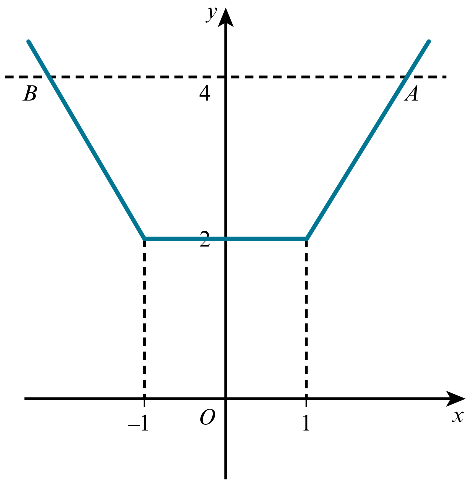
$-1 \leqslant x \leqslant 1$, so $x + 1 \geqslant 0$ but $x - 1 \leqslant 0$

$y = x + 1 - (x - 1)$
$= 2$

$x \leqslant -1$, so $x - 1 \leqslant 0$ and $x + 1 \leqslant 0$

$y = -(x + 1) - (x - 1)$
$= -x - 1 - x + 1$
$= -2x$

> The boundaries, 1 and -1, were chosen because these are the points at which the two terms contained within modulus symbols change from positive to negative or negative to positive.



b At point A: At point B:

$2x = 4$ $-2x = 4$

$x = 2$ $x = -2$

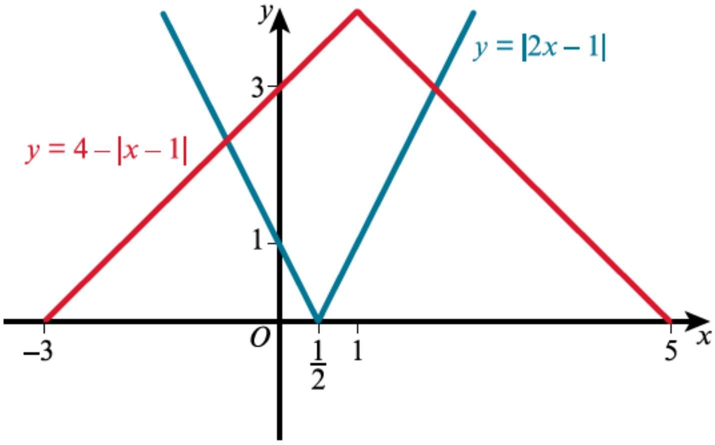
Solutions are: $x = 2$ or $x = -2$

EXERCISE 1C

1 Look for where the graph of $y = |x - 1|$ sits above the graph of $y = 2|x - 4|$. This is where the blue graph is above the red graph, i.e. $3 < x < 7$.

> A number of solutions refer to 'higher', 'above', etc. For any given value of x, the two graphs that you need to compare will pass through points with (usually) different y-coordinates. The graph with the larger y-coordinate at that point is the 'higher' graph.

2 **a**



b Graphs meet where

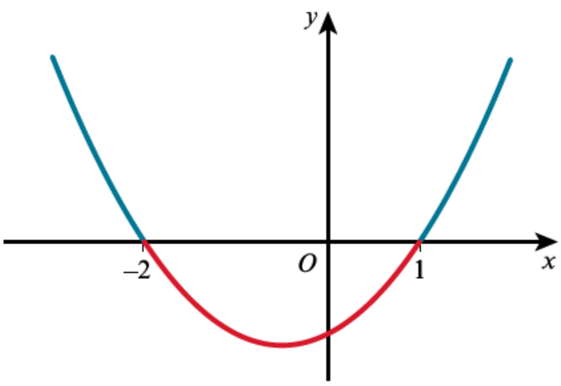
$$\begin{aligned} 2x - 1 &= 4 - (x - 1) \\ 2x - 1 &= 4 - x + 1 \\ 3x &= 6 \\ x &= 2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 1 - 2x &= 4 + (x - 1) \\ 1 - 2x &= 4 + x - 1 \\ 3x &= -2 \\ x &= -\frac{2}{3} \end{aligned}$$

$|2x + 1| > 4 - |x - 1|$ when $x > 2$ or $x < -\frac{2}{3}$.

3 **a** **Algebraic method**

$$\begin{aligned} |2x + 1| &\leqslant 3 \\ (2x + 1)^2 &\leqslant 9 \\ 4x^2 + 4x + 1 &\leqslant 9 \\ 4x^2 + 4x - 8 &\leqslant 0 \\ 4(x^2 + x - 2) &\leqslant 0 \\ 4(x + 2)(x - 1) &\leqslant 0 \end{aligned}$$

Critical values are -2 and 1.



Hence $-2 \leqslant x \leqslant 1$

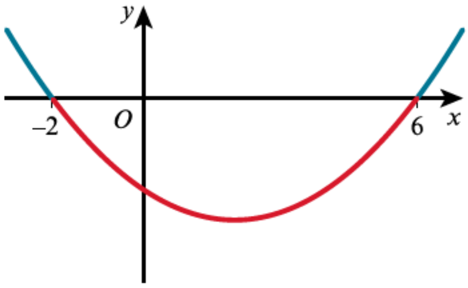
> Although this is an 'algebraic' method, you still need to use a method to decide whether your solution set sits between or outside of the critical values. If you are considering a quadratic, the

simplest way is to sketch its graph and look for the parts of the graph that sit above or below the x-axis.

b **Algebraic method**

$$|2-x|<4$$
$$(2-x)^2<16$$
$$4-4x+x^2<16$$
$$x^2-4x-12<0$$
$$(x-6)(x+2)<0$$

Critical values are 6 and -2.

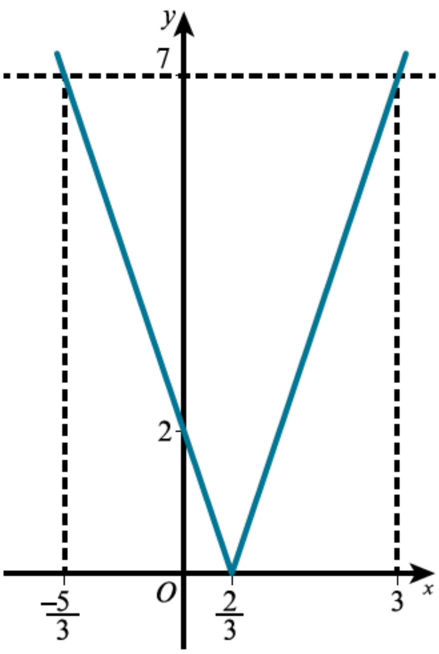


Hence $-2<x<6$.

c

For contrast, a graphical method is shown in this worked solution. Unless specifically instructed, you should choose whichever method you are most confident with.

Graphical method



$3x-2=7$ or $-(3x-2)=7$

$3x=9$ $\quad -3x+2=7$

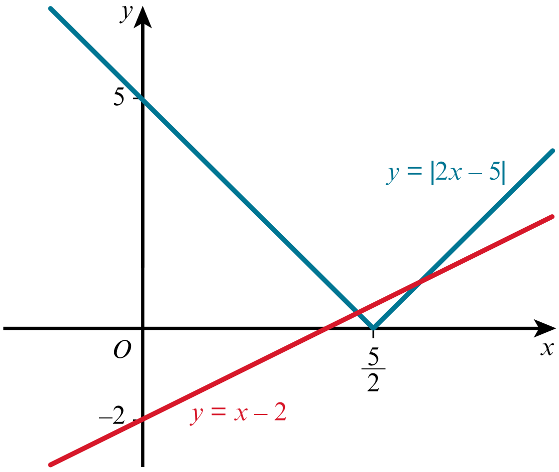
$x=3$ $\quad x=-\frac{5}{3}$

These are the critical values. Now consider where the graph of $y=|3x-2|$ lies above the line $y=7$.

$x\geqslant 3$ or $x\leqslant -\frac{5}{3}$

4 **a** **Graphical method**

$|2x-5|\leqslant x-2$



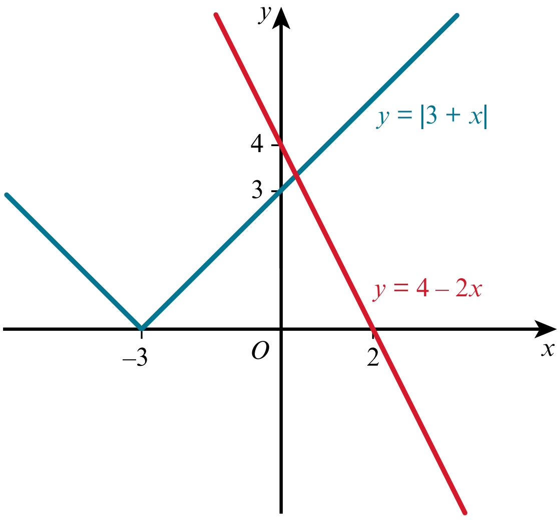
$$2x - 5 = x - 2 \qquad \text{or} \qquad -(2x - 5) = x - 2$$
$$x = 3 \qquad\qquad -2x + 5 = x - 2$$
$$3x = 7$$
$$x = \frac{7}{3}$$

These are the critical values. Now consider where the graph of $y = |2x - 5|$ lies underneath the line $y = x - 2$.

$\frac{7}{3} \leqslant x \leqslant 3$

b **Graphical method**

$|3 + x| > 4 - 2x$

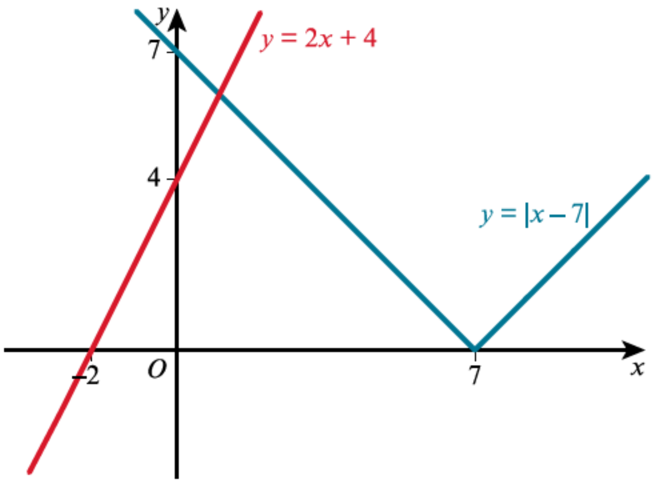


The graphs only meet once and that is where

$$3 + x = 4 - 2x$$
$$3x = 1$$
$$x = \frac{1}{3}$$

The graph of $y = |3 + x|$ lies above the line $y = 4 - 2x$ when $x > \frac{1}{3}$.

c **Graphical method**



$|x-7|-2x \leqslant 4$
$|x-7| \leqslant 2x+4$
$-(x-7) = 2x+4$
$-x+7 = 2x+4$
$3x = 3$
$x = 1$

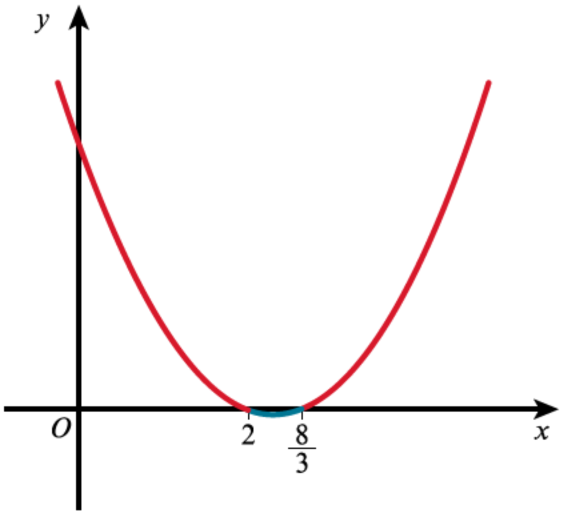
So the graph of $y=|x-7|$ lies below the line $y=2x+4$ when $x \geqslant 1$.

Here you will notice that $y=2x+4$ only intersects with the 'reflected' part of $y=|x-7|$, which is why the negative is introduced at the point of removing the modulus.

5 b **Algebraic solution**

$|2x-5| > |3-x|$
$(2x-5)^2 > (3-x)^2$
$4x^2-20x+25 > 9-6x+x^2$
$3x^2-14x+16 > 0$
$(3x-8)(x-2) > 0$

Critical values are 2 and $\frac{8}{3}$.

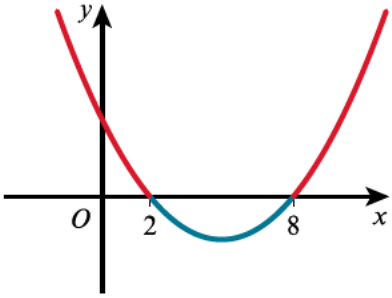


Hence $x < 2$ or $x > \frac{8}{3}$.

e **Algebraic solution**

$$3|3 - x| \geqslant |2x - 1|$$
$$9(3 - x)^2 \geqslant (2x - 1)^2$$
$$9(9 - 6x + x^2) \geqslant 4x^2 - 4x + 1$$
$$81 - 54x + 9x^2 \geqslant 4x^2 - 4x + 1$$
$$5x^2 - 50x + 80 \geqslant 0$$
$$x^2 - 10x + 16 \geqslant 0$$
$$(x - 8)(x - 2) \geqslant 0$$

Critical values are 2 and 8.



Hence $x \leqslant 2$ or $x \geqslant 8$.

A common error, when squaring products like the left–hand side of this inequality, is to leave the constant coefficient unchanged. You need to square everything.

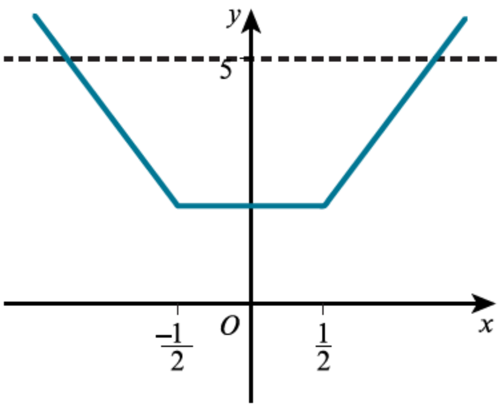
6 Draw the graph of $y = |2x + 1| + |2x - 1|$ by considering the points at which $2x + 1$ and $2x - 1$ change from being positive to negative, i.e. at $x = \pm\frac{1}{2}$.

$x > \frac{1}{2} \Rightarrow y = 2x + 1 + 2x - 1 = 4x$

$-\frac{1}{2} \leqslant x \leqslant \frac{1}{2} \Rightarrow y = 2x + 1 - (2x - 1) = 2x + 1 - 2x + 1 = 2$

$x < -\frac{1}{2} \Rightarrow y = -(2x + 1) - (2x - 1) = -2x - 2 - 2x + 2 = -4x$

The graph of $y = |2x + 1| + |2x - 1|$ is:



Critical values are $\frac{5}{4}$ and $-\frac{5}{4}$.

So either

$4x > 5$ or $-4x > 5$

$x > \frac{5}{4}$ or $x < -\frac{5}{4}$

EXERCISE 1D

1 **a**

$$
\begin{array}{r}
x^2 + 3x - 1 \\
x + 2 \overline{\smash{\big)} x^3 + 5x^2 + 5x - 2} \\
\underline{x^3 + 2x^2} \qquad\qquad \\
3x^2 + 5x \qquad \\
\underline{3x^2 + 6x} \qquad \\
-x - 2 \\
\underline{-x - 2} \\
0
\end{array}
$$

So

$(x^3 + 5x^2 + 5x - 2) \div (x + 2) = x^2 + 3x - 1$

e

$$
\begin{array}{r}
-5x^2 + 3x - 4 \\
2 - x \overline{\smash{\big)} 5x^3 - 13x^2 + 10x - 8} \\
\underline{5x^3 - 10x^2} \qquad\qquad \\
-3x^2 + 10x \qquad \\
\underline{-3x^2 + 6x} \qquad \\
4x - 8 \\
\underline{4x - 8} \\
0
\end{array}
$$

So

$(5x^3 - 13x^2 + 10x - 8) \div (2 - x) = -5x^2 + 3x - 4$

> Note that the x term in the divisor is negative in part **e**. You will find it helpful to think of $-x + 2$ instead of $2 - x$.

2 **c**

$$
\begin{array}{r}
4x^2 + \frac{5}{2}x + \frac{1}{4} \\
2x - 1 \overline{\smash{\big)} 8x^3 + x^2 - 2x + 1} \\
\underline{8x^3 - 4x^2} \qquad\qquad \\
5x^2 - 2x \qquad \\
\underline{5x^2 - \frac{5}{2}x} \qquad \\
\frac{1}{2}x + 1 \\
\underline{\frac{1}{2}x - \frac{1}{4}} \\
\frac{5}{4}
\end{array}
$$

So

Quotient $= 4x^2 + \frac{5}{2}x + \frac{1}{4}$

Remainder $= \frac{5}{4}$

> When dealing with algebraic division it is always best to **avoid** using decimals instead of fractions.

f

$$
\begin{array}{r}
5x^2 - 7 \\
x^2 + 0x + 1 \overline{\smash{\big)} 5x^4 + 0x^3 - 2x^2 - 13x + 8}
\end{array}
$$

$$\begin{array}{l} \underline{5x^4 + 0x^3 + 5x^2} \\ \qquad\quad -7x^2 - 13x + 8 \\ \qquad\quad \underline{-7x^2 + 0 - 7} \\ \qquad\qquad\quad -13x + 15 \end{array}$$

So

Quotient $= 5x^2 - 7$

Remainder $= -13x + 15$

In part **f** both the linear and quartic expressions have a 'missing' term. It is very important to keep the various sections of algebra lined up correctly, so use zeros as place holders for the missing terms.

3 For both parts **a** and **b** of Question **3** treat x as the variable and y as a constant.

a

$$\begin{array}{r|l} & x^2 + xy + y^2 \\ x - y & x^3 - y^3 \\ & \underline{x^3 - x^2y} \\ & x^2y - y^3 \\ & \underline{x^2y - xy^2} \\ & xy^2 - y^3 \\ & \underline{xy^2 - y^3} \\ & 0 \end{array}$$

So

$$\frac{x^3 - y^3}{x - y} = x^2 + xy + y^2$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

b

$$\begin{array}{r|l} & x^2 - xy + y^2 \\ x + y & x^3 + y^3 \\ & \underline{x^3 + x^2y} \\ & y^3 - x^2y \\ & \underline{-xy^2 - x^2y} \\ & y^3 + xy^2 \\ & \underline{y^3 + xy^2} \\ & 0 \end{array}$$

So

$$\frac{x^3 + y^3}{x + y} = x^2 - xy + y^2$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

4 **a**

$$\begin{array}{r|l} & 2x^2 + 11x - 21 \\ x - 2 & 2x^3 + 7x^2 - 43x + 42 \\ & \underline{2x^3 - 4x^2} \\ & 11x^2 - 43x \\ & \underline{11x^2 - 22x} \\ & -21x + 42 \\ & \underline{-21x + 42} \\ & 0 \end{array}$$

The remainder is zero, so $x - 2$ is a factor of $2x^3 + 7x^2 - 43x + 42$.

b

$$\begin{aligned} 2x^3 + 7x^2 - 43x + 42 &= (x - 2)(2x^2 + 11x - 21) \\ &= (x - 2)(2x - 3)(x + 7) \end{aligned}$$

5 **a**

$$\begin{array}{r} 6x^2 + 5x - 4 \\ 2x+1 \overline{)\,12x^3 + 16x^2 - 3x - 4} \\ \underline{12x^3 + 6x^2} \qquad\qquad \\ 10x^2 - 3x \quad\; \\ \underline{10x^2 + 5x} \quad\; \\ -8x - 4 \\ \underline{-8x - 4} \\ 0 \end{array}$$

The remainder is zero, so $2x + 1$ is a factor of $12x^3 + 16x^2 - 3x - 4$.

b $12x^3 + 16x^2 - 3x - 4 = (2x + 1)(6x^2 + 5x - 4)$

$= (2x + 1)(3x + 4)(2x - 1)$

So if

$12x^3 + 16x^2 - 3x - 4 = 0$

then

$(2x + 1)(3x + 4)(2x - 1) = 0$

So the solutions are:

$x = -\frac{1}{2}, x = -\frac{4}{3}, x = \frac{1}{2}$

6 **a**

$$\begin{array}{r} x^2 - 2x + 4 \\ x+1 \overline{)\,x^3 - x^2 + 2x + 4} \\ \underline{x^3 + x^2} \qquad\qquad \\ -2x^2 + 2x \quad\; \\ \underline{-2x^2 - 2x} \quad\; \\ 4x + 4 \\ \underline{4x + 4} \\ 0 \end{array}$$

The remainder is zero, so $x + 1$ is a factor of $x^3 - x^2 + 2x + 4$.

b $x^3 - x^2 + 2x + 4 = 0$

$(x + 1)(x^2 - 2x + 4) = 0$

$x = -1$ or $x^2 - 2x + 4 = 0$

For this last quadratic:

$b^2 - 4ac = (-2)^2 - 4 \times 1 \times 4 = 4 - 16 < 0$

The quadratic has no real solutions, so $x = -1$ is the only real root of $x^3 - x^2 + 2x + 4 = 0$.

Remember that the **discriminant** of the quadratic expression, $b^2 - 4ac$, reveals what type of roots the equation has.

- If the discriminant is positive, the quadratic has two distinct real roots.
- If the discriminant is zero, the quadratic has a repeated real root.
- If the discriminant is negative, the quadratic has no real roots.

7

There are many sources for this information and the equation is often presented in different ways. One example is shown in this worked solution.

If $ax^3 + bx^2 + cx + d = 0$, then:

$x = \left\{q + [q^2 + (r - p^2)^3]^{\frac{1}{2}}\right\}^{\frac{1}{3}} + \left\{q - [q^2 + (r - p^2)^3]^{\frac{1}{2}}\right\}^{\frac{1}{3}} + p$ ..... [1]

where

$p = -\frac{b}{3a}, q = p^3 + \frac{(bc - 3ad)}{6a^2}, r = \frac{c}{3a}$

In this specific case:

$2x^3 - 5x^2 - 28x + 15 = 0$

So

$a = 2,\ b = -5, c = -28,\ d = 15$

$$p = \frac{-(-5)}{3 \times 2} = \frac{5}{6}$$

$$\begin{aligned} q &= \left(\frac{5}{6}\right)^3 + \frac{5 \times 28 - 3 \times 2 \times 15}{6 \times 4} \\ &= \left(\frac{5}{6}\right)^3 + \frac{50}{24} \\ &= \frac{575}{216} \end{aligned}$$

$$r = -\frac{28}{3 \times 2} = -\frac{14}{3}$$

Check that your calculator gives the answer 5 when you substitute this into [1].

By the factor theorem this tells you that $x - 5$ is a factor of $2x^3 - 5x^2 - 28x + 15$.

Dividing $2x^3 - 5x^2 - 28x + 15$ by $x - 5$:

$$\begin{array}{r l} & 2x^2 + 5x - 3 \\ x - 5 \,\big) & \overline{2x^3 - 5x^2 - 28x + 15} \\ & \underline{2x^3 - 10x^2} \\ & \quad 5x^2 - 28x \\ & \quad \underline{5x^2 - 25x} \\ & \qquad -3x + 15 \\ & \qquad \underline{-3x + 15} \\ & \qquad\qquad 0 \end{array}$$

$$\begin{aligned} 2x^3 - 5x^2 - 28x + 15 &= (x - 5)(2x^2 + 5x - 3) \\ &= (x - 5)(2x - 1)(x + 3) \end{aligned}$$

If

$2x^3 - 5x^2 - 28x + 15 = 0$ then $(x - 5)(2x - 1)(x + 3) = 0$

Solutions are:

$x = 5, x = \frac{1}{2}, x = -3$

EXERCISE 1E

1 Let $f(x) = x^4 - 3x^3 - 4x^2 + 5x + 5$.

If $f(-1) = 0$, then $x + 1$ is a factor of $f(x)$.

$$\begin{aligned} f(-1) &= (-1)^4 - 3(-1)^3 - 4(-1)^2 + 5(-1) + 5 \\ &= 1 + 3 - 4 - 5 + 5 \\ &= 0 \end{aligned}$$

Hence $x + 1$ is a factor of $x^4 - 3x^3 - 4x^2 + 5x + 5$ by the factor theorem.

It's always helpful to call the original polynomial something like $f(x)$ or $g(x)$ because this saves you having to write it out repeatedly. Always state this clearly at the start of your solution.

2 Let $f(x) = 2x^3 - 7x^2 + 9x - 10$.

If $f\left(\frac{5}{2}\right) = 0$, then $2x - 5$ is a factor of $f(x)$.

$$\begin{aligned} f\left(\frac{5}{2}\right) &= 2\left(\frac{5}{2}\right)^3 - 7\left(\frac{5}{2}\right)^2 + 9\left(\frac{5}{2}\right) - 10 \\ &= 2\left(\frac{125}{8}\right) - 7\left(\frac{25}{4}\right) + 9\left(\frac{5}{2}\right) - 10 \\ &= \frac{250}{8} - \frac{350}{8} + \frac{180}{8} - \frac{80}{8} \\ &= 0 \end{aligned}$$

Hence $2x - 5$ is a factor of $2x^3 - 7x^2 + 9x - 10$ by the factor theorem.

3 $x + 4$ is a factor of $f(x) = x^3 + ax^2 - 29x + 12$.

So, by the factor theorem, $f(-4) = 0$.

$$\begin{aligned} (-4)^3 + a(-4)^2 - 29(-4) + 12 &= 0 \\ -64 + 16a + 116 + 12 &= 0 \\ 16a + 64 &= 0 \\ 16a &= -64 \\ a &= -\frac{64}{16} = -4 \end{aligned}$$

4 $x - 3$ is a factor of $f(x) = x^3 + ax^2 + bx - 30$.

So, by the factor theorem, $f(3) = 0$.

$$\begin{aligned} (3)^3 + a(3)^2 + b(3) - 30 &= 0 \\ 27 + 9a + 3b - 30 &= 0 \\ 9a + 3b - 3 &= 0 \\ 9a &= 3 - 3b \\ a &= \frac{3 - 3b}{9} = \frac{1 - b}{3} \end{aligned}$$

5 Let $f(x) = 2x^3 - x^2 + ax + b$.

Factorising $2x^2 + x - 1$:

$2x^2 + x - 1 = (2x - 1)(x + 1)$

So, if $2x^2 + x - 1$ is a factor of $f(x)$, then $2x - 1$ and $x + 1$ are also factors of $f(x)$.

If $2x - 1$ is a factor of $f(x)$, then, by the factor theorem, $f\left(\frac{1}{2}\right) = 0$.

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2 + a\left(\frac{1}{2}\right) + b = 0$$

$$\frac{1}{4} - \frac{1}{4} + \frac{a}{2} + b = 0$$

$$\frac{a}{2} + b = 0$$

$$a = -2b \text{ .... [1]}$$

Similarly, if $x + 1$ is a factor of $f(x)$, then $f(-1) = 0$.

$$f(-1) = 2(-1)^3 - (-1)^2 + a(-1) + b = 0$$

$$-2 - 1 - a + b = 0$$

$$b = a + 3 \text{ .... [2]}$$

Substituting [1] into [2]:

$$b = (-2b) + 3$$

$$3b = 3$$

$$b = 1$$

Then equation [1] gives $a = -2(1) = -2$.

When dealing with two different cases, make sure that you separate them clearly. If both result in equations that you need later, number the equations clearly, so that you can refer to them. Here the equations have been numbered [1] and [2].

6 Let $f(x) = 2x^3 + px^2 + (2q - 1)x + q$.

a If $x - 3$ is a factor of $f(x)$ then, by the factor theorem, $f(3) = 0$.

$$f(3) = 0$$

$$2(3)^3 + p(3)^2 + (2q - 1)(3) + q = 0$$

$$54 + 9p + 6q - 3 + q = 0$$

$$9p + 7q = -51 \text{ .... [1]}$$

Similarly, if $2x + 1$ is a factor of $f(x)$ then, by the factor theorem, $f\left(-\frac{1}{2}\right) = 0$.

$$f\left(-\frac{1}{2}\right) = 0$$

$$2\left(-\frac{1}{2}\right)^3 + p\left(-\frac{1}{2}\right)^2 + (2q - 1)\left(-\frac{1}{2}\right) + q = 0$$

$$-\frac{1}{4} + \frac{p}{4} - q + \frac{1}{2} + q = 0$$

$$\frac{p}{4} + \frac{1}{4} = 0$$

$$p = -1 \text{ .... [2]}$$

Substituting [2] into [1]:

$$9(-1) + 7q = -51$$

$$7q = -42$$

$$q = -6$$

b Substituting the values for p and q from part **a** into $f(x)$:

$$f(x) = 2x^3 + px^2 + (2q - 1)x + q$$

$$= 2x^3 - x^2 - 13x - 6$$

$$f(-2) = 2(-2)^3 - (-2)^2 - 13(-2) - 6$$

$$= -16 - 4 + 26 - 6$$

$$= 0$$

$f(-2) = 0$, so $x + 2$ is also a factor of the expression, by the factor theorem.

When you are asked to 'verify' that an expression is a factor of $f(x)$ it is fine to use the factor theorem. You do not need to use long division unless you need to go on to factorise the entire expression.

7 Let $\mathrm{f}(x) = x^3 + 4x^2 + 7ax + 4a$.

If $x + a$ is a factor of $\mathrm{f}(x)$ then, by the factor theorem, $\mathrm{f}(-a) = 0$.

$$\begin{aligned} \mathrm{f}(-a) &= 0 \\ (-a)^3 + 4(-a)^2 + 7a(-a) + 4a &= 0 \\ -a^3 + 4a^2 - 7a^2 + 4a &= 0 \\ a^3 + 3a^2 - 4a &= 0 \\ a(a^2 + 3a - 4) &= 0 \\ a = 0 \text{ or } a^2 + 3a - 4 &= 0 \\ a = 0 \text{ or } (a+4)(a-1) &= 0 \\ a &= 0, -4 \text{ or } 1 \end{aligned}$$

8 Let $\mathrm{f}(x) = x^3 + px + q$ and $\mathrm{g}(x) = x^3 + (1-p)x^2 + 19x - 2q$.

a If $x + 1$ is a factor of $\mathrm{f}(x)$ then, by the factor theorem, $\mathrm{f}(-1) = 0$.

$$\begin{aligned} \mathrm{f}(-1) &= 0 \\ (-1)^3 + p(-1) + q &= 0 \\ -1 - p + q &= 0 \\ q &= p + 1 \text{ .... [1]} \end{aligned}$$

Similarly, if $x + 1$ is a factor of $\mathrm{g}(x)$ then, by the factor theorem, $\mathrm{g}(-1) = 0$

$$\begin{aligned} \mathrm{g}(-1) &= 0 \\ (-1)^3 + (1-p)(-1)^2 + 19(-1) - 2q &= 0 \\ -1 + 1 - p - 19 - 2q &= 0 \\ p + 2q &= -19 \text{ ..... [2]} \end{aligned}$$

Substituting [1] into [2]:

$$\begin{aligned} p + 2(p+1) &= -19 \\ p + 2p + 2 &= -19 \\ 3p &= -21 \\ p &= -7 \end{aligned}$$

Substituting $p = -7$ into [1]: $q = -7 + 1 = -6$

b Substituting the values for p and q from part **a** into $\mathrm{f}(x)$ and $\mathrm{g}(x)$:

$\mathrm{f}(x) = x^3 - 7x - 6$ and $\mathrm{g}(x) = x^3 + 8x^2 + 19x + 12$

Using long division to divide $\mathrm{f}(x)$ by $x + 1$:

$$\begin{array}{r} x^2 - x - 6 \\ x+1 \overline{) x^3 + 0x^2 - 7x - 6} \\ \underline{x^3 + x^2} \qquad\qquad \\ -x^2 - 7x \qquad \\ \underline{-x^2 - x} \qquad \\ -6x - 6 \\ \underline{-6x - 6} \\ 0 \end{array}$$

So

$$\begin{aligned} \mathrm{f}(x) &= (x+1)(x^2 - x - 6) \\ &= (x+1)(x-3)(x+2) \end{aligned}$$

Using long division to divide $\mathrm{g}(x)$ by $x + 1$:

$$\begin{array}{r} x^2 + 7x + 12 \\ x+1 \overline{) x^3 + 8x^2 + 19x + 12} \\ \underline{x^3 + x^2} \qquad\qquad\qquad \\ 7x^2 + 19x \qquad \\ \underline{7x^2 + 7x} \qquad \\ 12x + 12 \\ \underline{12x + 12} \\ 0 \end{array}$$

So

$$\begin{aligned} g(x) &= (x+1)(x^2+7x+12) \\ &= (x+1)(x+3)(x+4) \end{aligned}$$

> You might need to divide polynomials very quickly. There are several methods, but make sure that you are very comfortable with at least one. Long division is the most commonly seen method in textbooks and sample solutions.

9 Let $f(x) = x^4 - x^3 + px^2 - 11x + q$.

a If $x-1$ is a factor of $f(x)$ then, by the factor theorem, $f(1) = 0$.

$$\begin{aligned} f(1) &= 0 \\ 1 - 1 + p - 11 + q &= 0 \\ p + q &= 11 \text{ .... [1]} \end{aligned}$$

Similarly, if $x+2$ is a factor of $f(x)$ then, by the factor theorem, $f(-2) = 0$.

$$\begin{aligned} f(-2) &= 0 \\ (-2)^4 - (-2)^3 + p(-2)^2 - 11(-2) + q &= 0 \\ 16 + 8 + 4p + 22 + q &= 0 \\ 4p + q &= -46 \text{ .... [2]} \end{aligned}$$

Considering the pair of simultaneous equations:

$4p + q = -46$ ...... [2]

$p + q = 11$ .... [1]

[2] − [1]:

$$\begin{aligned} 3p &= -57 \\ p &= -19 \\ q &= 11 - p = 11 - (-19) = 30 \end{aligned}$$

b Substituting the values for p and q from part **a** into $f(x)$:

$f(x) = x^4 - x^3 - 19x^2 - 11x + 30$

Since $f(x)$ is divisible by $x-1$, using long division to divide $f(x)$ by $x-1$:

$$\begin{array}{r} x^3 + 0x^2 - 19x - 30 \\ x-1 \,\big)\, \overline{x^4 - x^3 - 19x^2 - 11x + 30} \\ \underline{x^4 - x^3} \\ 0x^3 - 19x^2 \\ \underline{0x^3 + 0x^2} \\ -19x^2 - 11x \\ \underline{-19x^2 + 19x} \\ -30x + 30 \\ \underline{-30x + 30} \\ 0 \end{array}$$

So $f(x) = (x-1)(x^3 - 19x - 30)$

$f(x)$ is also divisible by $x+2$, but $x-1$ is not divisible by $x+2$, so $x^3 - 19x - 30$ must be divisible by $x+2$.

Using long division to divide $x^3 - 19x - 30$ by $x+2$:

$$\begin{array}{r} x^2 - 2x - 15 \\ x+2 \,\big)\, \overline{x^3 + 0x^2 - 19x - 30} \\ \underline{x^3 + 2x^2} \\ -2x^2 - 19x \\ \underline{-2x^2 - 4x} \\ -15x - 30 \\ \underline{-15x - 30} \\ 0 \end{array}$$

So

$$\begin{aligned}f(x) &= (x-1)(x^3-19x-30)\\ &= (x-1)(x+2)(x^2-2x-15)\\ &= (x-1)(x+2)(x-5)(x+3)\end{aligned}$$

If the degree of the term you are dividing by is more than one fewer than the polynomial, you are likely to need to complete more than one division in your solution.

10 In all of the parts in Question **10** you need to 'spot' a root, so that you can find the first factor. Unless the polynomial is particularly complicated, you should expect that at least one factor of the constant term will work. You can see why this works if you multiply any number of brackets together – the constant term of the polynomial will turn out to be the product of the constant terms in each bracket (unless all of the brackets contain more than one x). Remember to try both positive and negative numbers. In each solution shown here different numbers and, hence, different factors could have been used as the starting point.

a Let $f(x) = x^3 - 5x^2 - 4x + 20$.

The positive and negative factors of 20 are $\pm1, \pm2, \pm4, \pm5, \pm10$ and ±20.

$$\begin{aligned}f(x) &= x^3 - 5x^2 - 4x + 20\\ f(2) &= (2)^3 - 5(2)^2 - 4(2) + 20\\ &= 8 - 20 - 8 + 20 = 0\end{aligned}$$

$f(2) = 0$, so $x - 2$ is a factor of $f(x)$ by the factor theorem.

Using long division to divide $f(x)$ by $x - 2$:

$$\begin{array}{r}x^2 - 3x - 10\\ x-2\overline{\smash{)}\,x^3 - 5x^2 - 4x + 20}\\ \underline{x^3 - 2x^2}\qquad\qquad\quad\\ -3x^2 - 4x\qquad\quad\\ \underline{-3x^2 + 6x}\qquad\quad\\ -10x + 20\\ \underline{-10x + 20}\\ 0\end{array}$$

$$\begin{aligned}f(x) &= x^3 - 5x^2 - 4x + 20\\ &= (x-2)(x^2 - 3x - 10)\\ &= (x-2)(x-5)(x+2)\end{aligned}$$

Setting $f(x) = 0$:

$(x-2)(x-5)(x+2) = 0$

$x = 2, 5$ or -2

b Let $f(x) = x^3 + 5x^2 - 17x - 21$.

The positive and negative factors of -21 are $\pm1, \pm3, \pm7$ and ±21.

$$\begin{aligned}f(x) &= x^3 + 5x^2 - 17x - 21\\ f(-1) &= (-1)^3 + 5(-1)^2 - 17(-1) - 21\\ &= -1 + 5 + 17 - 21 = 0\end{aligned}$$

$f(-1) = 0$, so $x + 1$ is a factor of $f(x)$ by the factor theorem.

Using long division to divide $f(x)$ by $x + 1$:

$$\begin{array}{r}x^2 + 4x - 21\\ x+1\overline{\smash{)}\,x^3 + 5x^2 - 17x - 21}\\ \underline{x^3 + x^2}\qquad\qquad\quad\\ 4x^2 - 17x\qquad\\ \underline{4x^2 + 4x}\qquad\\ -21x - 21\\ \underline{-21x - 21}\\ 0\end{array}$$

$$\begin{aligned}f(x) &= x^3+5x^2-17x-21\\ &= (x+1)(x^2+4x-21)\\ &= (x+1)(x+7)(x-3)\end{aligned}$$

Setting $f(x)=0$:

$(x+1)(x+7)(x-3)=0$

$x=-1, -7$ or 3

c Let $f(x)=2x^3-5x^2-13x+30$.

The positive and negative factors of 30 are $\pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15$ and ±30.

$$\begin{aligned}f(x) &= 2x^3-5x^2-13x+30\\ f(2) &= 2(2)^3-5(2)^2-13(2)+30\\ &= 16-20-26+30=0\end{aligned}$$

$f(2)=0$, so $x-2$ is a factor of $f(x)$ by the factor theorem.

Using long division to divide $f(x)$ by $x-2$:

$$\begin{array}{r} 2x^2-x-15 \\ x-2\,\overline{)\,2x^3-5x^2-13x+30} \\ \underline{2x^3-4x^2} \qquad\qquad \\ -x^2-13x \qquad \\ \underline{-x^2+2x} \qquad \\ -15x+30 \\ \underline{-15x+30} \\ 0 \end{array}$$

$$\begin{aligned}f(x) &= 2x^3-5x^2-13x+30\\ &= (x-2)(2x^2-x-15)\\ &= (x-2)(2x+5)(x-3)\end{aligned}$$

Setting $f(x)=0$:

$(x-2)(2x+5)(x-3)=0$

$x=2, -\frac{5}{2}$ or 3

d Let $f(x)=3x^3+17x^2+18x-8$.

The positive and negative factors of -8 are $\pm1, \pm2, \pm4$ and ±8.

$$\begin{aligned}f(x) &= 3x^3+17x^2+18x-8\\ f(-2) &= 3(-2)^3+17(-2)^2+18(-2)-8\\ &= -24+68-36-8=0\end{aligned}$$

$f(-2)=0$, so $x+2$ is a factor of $f(x)$ by the factor theorem.

Using long division to divide $f(x)$ by $x+2$:

$$\begin{array}{r} 3x^2+11x-4 \\ x+2\,\overline{)\,3x^3+17x^2+18x-8} \\ \underline{3x^3+6x^2} \qquad\qquad \\ 11x^2+18x \qquad \\ \underline{11x^2+22x} \qquad \\ -4x-8 \\ \underline{-4x-8} \\ 0 \end{array}$$

$$\begin{aligned}f(x) &= 3x^3+17x^2+18x-8\\ &= (x+2)(3x^2+11x-4)\\ &= (x+2)(3x-1)(x+4)\end{aligned}$$

Setting $f(x)=0$:

$(x+2)(3x-1)(x+4)=0$

$x=-2, \frac{1}{3}$ or -4

e Let $\mathrm{f}(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$.

The positive and negative factors of 12 are $\pm1, \pm2, \pm3, \pm4, \pm6$ and ±12.

$$\begin{aligned}\mathrm{f}(x) &= x^4 + 2x^3 - 7x^2 - 8x + 12\\ \mathrm{f}(1) &= (1)^4 + 2(1)^3 - 7(1)^2 - 8(1) + 12\\ &= 1 + 2 - 7 - 8 + 12 = 0\end{aligned}$$

$\mathrm{f}(1) = 0$, so $x - 1$ is a factor of $\mathrm{f}(x)$ by the factor theorem.

Using long division to divide $\mathrm{f}(x)$ by $x - 1$:

$$\begin{array}{r} x^3 + 3x^2 - 4x - 12 \\ x - 1\overline{)x^4 + 2x^3 - 7x^2 - 8x + 12} \\ \underline{x^4 - x^3} \qquad\qquad\qquad\quad \\ 3x^3 - 7x^2 \qquad\qquad \\ \underline{3x^3 - 3x^2} \qquad\qquad \\ -4x^2 - 8x \qquad \\ \underline{-4x^2 + 4x} \qquad \\ -12x + 12 \\ \underline{-12x + 12} \\ 0 \end{array}$$

$$\begin{aligned}\mathrm{f}(x) &= x^4 + 2x^3 - 7x^2 - 8x + 12\\ &= (x - 1)(x^3 + 3x^2 - 4x - 12)\end{aligned}$$

Let $\mathrm{g}(x) = x^3 + 3x^2 - 4x - 12$.

The positive and negative factors of -12 are $\pm1, \pm2, \pm3, \pm4, \pm6$ and ±12.

$$\begin{aligned}\mathrm{g}(x) &= x^3 + 3x^2 - 4x - 12\\ \mathrm{g}(2) &= (2)^3 + 3(2)^2 - 4(2) - 12\\ &= 8 + 12 - 8 - 12 = 0\end{aligned}$$

$\mathrm{g}(2) = 0$, so $x - 2$ is a factor of $\mathrm{g}(x)$ by the factor theorem.

Using long division to divide $\mathrm{g}(x)$ by $x - 2$:

$$\begin{array}{r} x^2 + 5x + 6 \\ x - 2\overline{)x^3 - 3x^2 - 4x - 12} \\ \underline{x^3 - 2x^2} \qquad\qquad\quad \\ 5x^2 - 4x \qquad \\ \underline{5x^2 - 10x} \qquad \\ 6x - 12 \\ \underline{6x - 12} \\ 0 \end{array}$$

$$\begin{aligned}\mathrm{g}(x) &= x^3 + 3x^2 - 4x - 12\\ &= (x - 2)(x^2 + 5x + 6)\\ &= (x - 2)(x + 3)(x + 2)\end{aligned}$$

So

$$\begin{aligned}\mathrm{f}(x) &= (x - 1)(x^3 + 3x^2 - 4x - 12)\\ &= (x - 1)(x - 2)(x + 3)(x + 2)\end{aligned}$$

Setting $\mathrm{f}(x) = 0$:

$(x - 1)(x - 2)(x + 3)(x + 2) = 0$

$x = 1,\ 2, -3$ or -2

f Let $\mathrm{f}(x) = 2x^4 - 11x^3 + 12x^2 + x - 4$.

The positive and negative factors of -4 are $\pm1, \pm2$ and ±4.

$$\begin{aligned}\mathrm{f}(x) &= 2x^4 - 11x^3 + 12x^2 + x - 4\\ \mathrm{f}(1) &= 2(1)^4 - 11(1)^3 + 12(1)^2 + (1) - 4\\ &= 2 - 11 + 12 + 1 - 4 = 0\end{aligned}$$

$\mathrm{f}(1) = 0$, so $x - 1$ is a factor of $\mathrm{f}(x)$ by the factor theorem.

Using long division to divide $\mathrm{f}(x)$ by $x - 1$:

$$
\begin{array}{r}
2x^3 - 9x^2 + 3x + 4 \\
x-1 \overline{\smash{)}\, 2x^4 - 11x^3 + 12x^2 + x - 4} \\
\underline{2x^4 - 2x^3} \qquad\qquad\qquad\qquad \\
-9x^3 + 12x^2 \qquad\qquad \\
\underline{-9x^3 + 9x^2} \qquad\qquad \\
3x^2 + x \qquad \\
\underline{3x^2 - 3x} \qquad \\
4x - 4 \\
\underline{4x - 4} \\
0
\end{array}
$$

$$
\begin{aligned}
f(x) &= 2x^4 - 11x^3 + 12x^2 + x - 4 \\
&= (x-1)(2x^3 - 9x^2 + 3x + 4)
\end{aligned}
$$

Let $g(x) = 2x^3 - 9x^2 + 3x + 4$.

$$
\begin{aligned}
g(x) &= 2x^3 - 9x^2 + 3x + 4 \\
g(1) &= 2(1)^3 - 9(1)^2 + 3(1) + 4 \\
&= 2 - 9 + 3 + 4 = 0
\end{aligned}
$$

$g(1) = 0$, so $x - 1$ is a factor of $g(x)$ by the factor theorem.

Using long division to divide $g(x)$ by $x - 1$:

$$
\begin{array}{r}
2x^2 - 7x - 4 \\
x-1 \overline{\smash{)}\, 2x^3 - 9x^2 + 3x + 4} \\
\underline{2x^3 - 2x^2} \qquad\qquad\qquad \\
-7x^2 + 3x \qquad\quad \\
\underline{-7x^2 + 7x} \qquad\quad \\
-4x + 4 \\
\underline{-4x + 4} \\
0
\end{array}
$$

$$
\begin{aligned}
g(x) &= 2x^3 - 9x^2 + 3x + 4 \\
&= (x-1)(2x^2 - 7x - 4) \\
&= (x-1)(2x+1)(x-4)
\end{aligned}
$$

So

$$
\begin{aligned}
f(x) &= (x-1)(2x^3 - 9x^2 + 3x + 4) \\
&= (x-1)(x-1)(2x+1)(x-4)
\end{aligned}
$$

Setting $f(x) = 0$:

$$(x-1)(x-1)(2x+1)(x-4) = 0$$

$$x = 1, -\frac{1}{2}, \text{ or } 4$$

11

In the worked solution shown here, 'f' has been used as the common difference instead of 'd'. You can use any letter, as long as you are consistent and as long as each letter has a distinct purpose.

Given that the roots are consecutive terms in an arithmetic progression, the roots can be written as $e - f$, e, and $e + f$. Because these are roots, by the factor theorem, the three factors of the polynomial will be $x - (e - f)$, $x - e$, and $x - (e + f)$, i.e. $x - e + f$, $x - e$, and $x - e - f$.

So

$$
\begin{aligned}
&x^3 + ax^2 + bx + c = (x - e + f)(x - e)(x - e - f) \\
&= (x - e)(x^2 - ex - fx - ex + e^2 + ef + fx - ef - f^2) \\
&= (x - e)(x^2 - 2ex + e^2 - f^2) \\
&= x^3 - 2ex^2 + e^2x - f^2x - ex^2 + 2e^2x - e^3 + ef^2 \\
&= x^3 + (-3e)x^2 + (3e^2 - f^2)x - e^3 + ef^2
\end{aligned}
$$

and

$$
\begin{aligned}
&a = -3e \\
&b = 3e^2 - f^2 \Rightarrow eb = 3e^3 - ef^2 \\
&c = -e^3 + ef^2
\end{aligned}
$$

$eb + c = 2e^3$

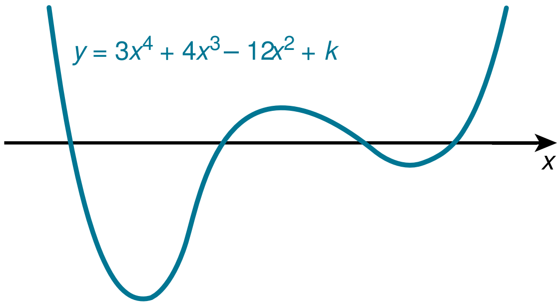
and $a^3 = -27e^3$

So $\dfrac{a^3}{eb+c} = -\dfrac{27}{2}$

$$2a^3 = -27(eb + c)$$

$$2a^3 + 27c = -27eb = 9(-3e)b = 9ab$$

12 Consider the graph with equation $y = 3x^4 + 4x^3 - 12x^2 + k$. If the equation has four real roots, then the graph will cross the x-axis four separate times. This means that the shape of the graph must look like this:



The y-coordinates of the two outer turning points, therefore, must be negative and the y-coordinate of the middle turning point must be positive.

Finding the turning points by setting $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^3 + 12x^2 - 24x = 0$$

$$x^3 + x^2 - 2x = 0$$

$$x(x^2 + x - 2) = 0$$

$$x(x - 1)(x + 2) = 0$$

$x = 1, 0$ or -2

When $x = -2$, $y = k - 32$

When $x = 0$, $y = k$

When $x = 1$, $y = k - 5$

So you need the outer two y-coordinates to be negative and the middle one positive,

i.e.

$k > 0$

and

$k - 5 < 0$ giving $k < 5$

and

$k - 32 < 0$ giving $k < 32$

So the set of values is

$0 < k < 5$

EXERCISE 1F

1 **a** Let $f(x) = 6x^3 + 3x^2 - 5x + 2$.

Substituting $x = 1$:

$$\begin{aligned}\text{Remainder} &= f(1)\\ &= 6(1)^3 + 3(1)^2 - 5(1) + 2\\ &= 6 + 3 - 5 + 2 = 6\end{aligned}$$

d Let $f(x) = 4x^3 - x^2 - 18x + 1$.

Substituting $x = \frac{1}{2}$:

$$\begin{aligned}\text{Remainder} &= 4\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2 - 18\left(\frac{1}{2}\right) + 1\\ &= \frac{4}{8} - \frac{1}{4} - \frac{18}{2} + 1\\ &= \frac{4 - 2 - 72 + 8}{8} = -\frac{62}{8} = -\frac{31}{4}\end{aligned}$$

2 **a** Let $f(x) = x^3 - 3x^2 + ax - 7$.

When $f(x)$ is divided by $x + 2$ the remainder is –37.

Hence, by the remainder theorem, $f(-2) = -37$.

$$\begin{aligned}f(-2) &= -37\\ (-2)^3 - 3(-2)^2 + a(-2) - 7 &= -37\\ -8 - 12 - 2a - 7 &= -37\\ 2a &= 10\\ a &= 5\end{aligned}$$

b Let $f(x) = 9x^3 + bx - 5$.

When $f(x)$ is divided by $3x + 2$ the remainder is -13.

Hence, by the remainder theorem, $f\left(-\frac{2}{3}\right) = -13$.

$$\begin{aligned}f\left(-\frac{2}{3}\right) &= -13\\ 9\left(-\frac{2}{3}\right)^3 + b\left(-\frac{2}{3}\right) - 5 &= -13\\ -\frac{9 \times 8}{27} - \frac{2b}{3} - 5 &= -13\\ -\frac{8}{3} - \frac{2b}{3} - 5 &= -13\\ -8 - 2b - 15 &= -39\\ 2b &= 16\\ b &= 8\end{aligned}$$

3 $x - 1$ is a factor of $f(x)$, so $f(1) = 0$ by the factor theorem.

$$\begin{aligned}(1)^3 + a(1)^2 + b(1) - 5 &= 0\\ 1 + a + b - 5 &= 0\\ a + b &= 4 \text{ .... [1]}\end{aligned}$$

When $f(x)$ is divided by $x + 1$ the remainder is -6.

Hence, by the remainder theorem, $f(-1) = -6$.

$$\begin{aligned}f(-1) &= -6\\ (-1)^3 + a(-1)^2 + b(-1) - 5 &= -6\\ -1 + a - b - 5 &= -6\\ a - b &= 0 \text{ .... [2]}\end{aligned}$$

[1] + [2]:

$2a = 4$
$a = 2$

Substituting into [1]:

$2 + b = 4$
$b = 2$

4 $x + 2$ is a factor of $f(x)$, so $f(-2) = 0$ by the factor theorem.

$$3(-2)^3 + a(-2)^2 + b(-2) + 8 = 0$$
$$-3 \times 8 + 4a - 2b + 8 = 0$$
$$4a - 2b = 24 - 8 = 16$$
$$2a - b = 8 \text{ .... [1]}$$

When $f(x)$ is divided by $x - 1$ the remainder is 15.

Hence, by the remainder theorem, $f(1) = 15$.

$$f(1) = 15$$
$$3(1)^3 + a(1)^2 + b(1) + 8 = 15$$
$$3 + a + b + 8 = 15$$
$$a + b = 4 \text{ .... [2]}$$

[1] + [2]:

$3a = 12$
$a = 4$

Substituting into [2]:

$4 + b = 4$
$b = 0$

5 **a** $2x + 1$ is a factor of $f(x)$, so $f\left(-\frac{1}{2}\right) = 0$ by the factor theorem.

$$a\left(-\frac{1}{2}\right)^3 + 7\left(-\frac{1}{2}\right)^2 + b\left(-\frac{1}{2}\right) - 8 = 0$$
$$-\frac{a}{8} + \frac{7}{4} - \frac{b}{2} - 8 = 0$$
$$-a + 14 - 4b - 64 = 0$$
$$a + 4b = -50 \text{ .... [1]}$$

When $f(x)$ is divided by $x + 1$ the remainder is 7

Hence, by the remainder theorem, $f(-1) = 7$.

$$f(-1) = 7$$
$$a(-1)^3 + 7(-1)^2 + b(-1) - 8 = 7$$
$$-a + 7 - b - 8 = 7$$
$$a + b = -8 \text{ .... [2]}$$

[1] − [2]:

$3b = -42$
$b = -14$

Substituting into [2]:

$a - 14 = -8$
$a = 6$

b $2x + 1$ is a factor of $f(x)$, so using long division to divide $f(x)$ by $2x + 1$:

$$\begin{array}{r} 3x^2 + 2x - 8 \\ 2x + 1 \overline{\smash{)}\, 6x^3 + 7x^2 - 14x - 8} \\ \underline{6x^3 + 3x^2} \qquad\qquad\quad \\ 4x^2 - 14x \qquad \\ \underline{4x^2 + 2x} \qquad \\ -16x - 8 \\ \underline{-16x - 8} \\ 0 \end{array}$$

So

$$6x^3 + 7x^2 - 14x - 8 = (2x+1)(3x^2+2x-8)$$
$$= (2x+1)(3x-4)(x+2)$$

6 Let $f(x) = x^3 + 6x^2 + px - 3$.

a When $f(x)$ is divided by $x+1$ the remainder is R.

Hence, by the remainder theorem, $f(-1) = R$.

$$f(-1) = R$$
$$(-1)^3 + 6(-1)^2 + p(-1) - 3 = R$$
$$-1 + 6 - p - 3 = R$$
$$p = 2 - R \text{ .... [1]}$$

When $f(x)$ is divided by $x-3$ the remainder is $-10R$.

Hence, by the remainder theorem, $f(3) = -10R$.

$$f(3) = -10R$$
$$(3)^3 + 6(3)^2 + p(3) - 3 = -10R$$
$$27 + 54 + 3p - 3 = -10R$$
$$3p = -10R - 78 \text{ .... [2]}$$

Substituting [1] into [2]:

$$3(2-R) = -10R - 78$$
$$6 - 3R = -10R - 78$$
$$7R = -84$$
$$R = -12$$

Substituting into [1]:

$$p = 2 - R$$
$$p = 2 - (-12) = 14$$

For simultaneous equations in two unknowns, you can always rearrange one equation to make either variable the subject and substitute the result into the second equation. This method works for all linear simultaneous equations, although mostly elimination by addition has been used in these worked solutions.

b $f(x) = x^3 + 6x^2 + 14x - 3$

By the remainder theorem, the remainder when $f(x)$ is divided by $x-2$ is $f(2)$.

$$f(2) = (2)^3 + 6(2)^2 + 14(2) - 3$$
$$= 8 + 24 + 28 - 3$$
$$= 57$$

7 **a** $x-2$ is a factor of $f(x)$, so $f(2) = 0$ by the factor theorem.

$$(2)^3 + a(2)^2 + b(2) + 2 = 0$$
$$8 + 4a + 2b + 2 = 0$$
$$4a + 2b = -10$$
$$2a + b = -5 \text{ .... [1]}$$

When $f(x)$ is divided by $x+1$ the remainder is 21.

Hence, by the remainder theorem, $f(-1) = 21$.

$$f(-1) = 21$$
$$(-1)^3 + a(-1)^2 + b(-1) + 2 = 21$$
$$-1 + a - b + 2 = 21$$
$$a - b = 20 \text{ .... [2]}$$

[1] + [2]:

$$3a = 15$$
$$a = 5$$

Substituting into [2]:

$$5 - b = 20$$
$$b = -15$$

b Substituting the values from part **a** for a and b:

$f(x) = x^3 + 5x^2 - 15x + 2$

$x - 2$ is a factor of $f(x)$, so using long division to divide $f(x)$ by $x - 2$:

$$\begin{array}{r} x^2 + 7x - 1 \\ x - 2 \overline{)\, x^3 + 5x^2 - 15x + 2} \\ \underline{x^3 - 2x^2} \qquad\qquad\quad \\ 7x^2 - 15x \qquad \\ \underline{7x^2 - 14x} \qquad \\ -x + 2 \\ \underline{-x + 2} \\ 0 \end{array}$$

So

$$x^3 + 5x^2 - 15x + 2 = (x - 2)(x^2 + 7x - 1)$$
$$(x - 2)(x^2 + 7x - 1) = 0$$
$$x = 2$$
$$\text{or } x = \frac{-7 \pm \sqrt{7^2 - 4 \times 1 \times -1}}{2 \times 1}$$
$$= \frac{-7 \pm \sqrt{49 + 4}}{2}$$
$$= \frac{-7 \pm \sqrt{53}}{2}$$

An alternative to using the quadratic formula is to complete the square by hand. Both methods are valid.

8 **a** Since $-1, 2$ and k are all roots of $f(x)$, by the factor theorem $f(-1) = f(2) = f(k) = 0$.

Substituting $x = -1$:

$$f(-1) = 0$$
$$2(-1)^3 + a(-1)^2 + b(-1) + c = 0$$
$$-2 + a - b + c = 0$$
$$a - b + c = 2 \text{ .... [1]}$$

Substituting $x = 2$:

$$f(2) = 0$$
$$2(2)^3 + a(2)^2 + b(2) + c = 0$$
$$16 + 4a + 2b + c = 0$$
$$4a + 2b + c = -16 \text{ .... [2]}$$

When $f(x)$ is divided by $x - 1$ the remainder is 6.

Hence, by the remainder theorem, $f(1) = 6$.

$$f(1) = 6$$
$$2(1)^3 + a(1)^2 + b(1) + c = 6$$
$$2 + a + b + c = 6$$
$$a + b + c = 4 \text{ .... [3]}$$

[2] − [1]:

$3a + 3b = -18$

$a + b = -6$ .... [4]

[2] − [3]:

$3a + b = -20$ .... [5]

[5] − [4]:

$$2a = -14$$
$$a = -7$$

Substituting into [4]:

$$-7 + b = -6$$
$$b = 1$$

Substituting into [3]:

$$-7 + 1 + c = 4$$
$$c = 10$$

There are several possible approaches to finding k from this point. The method first shown here uses a straightforward but longer approach. After that, a shorter, different method is shown.

$x = -1$ and $x = 2$ are roots of $f(x)$.

So $x + 1$ and $x - 2$ are both factors of $f(x)$.

Using long division to divide $f(x)$ by $x^2 - x - 2$:

$$\begin{array}{r} 2x - 5 \\ x^2 - x - 2 \overline{\smash{)}2x^3 - 7x^2 + x + 10} \\ \underline{2x^3 - 2x^2 - 4x} \\ -5x^2 + 5x + 10 \\ \underline{-5x^2 + 5x + 10} \\ 0 \end{array}$$

$$\begin{aligned} 2x^3 - 7x^2 + x + 10 &= (x^2 - x - 2)(2x - 5) \\ &= (x - 2)(x + 1)(2x - 5) \end{aligned}$$

Roots of $2x^3 - 7x^2 + x + 10$ are:

$x = 2, -1, \frac{5}{2}$

So $k = \frac{5}{2}$

Alternative method

You know that $f(x) = 2x^3 - 7x^2 + x + 10$, so $f(x) = 0$ is equivalent to

$2x^3 - 7x^2 + x + 10 = 0$

i.e. $x^3 - \frac{7}{2}x^2 + \frac{1}{2}x + 5 = 0$ ..... [1]

If the roots of this are $x = 2, -1, k$, then $(x - 2)(x + 1)(x - k) = 0$.

This is equivalent to [1].

Expansion of this triple bracket gives a constant term of $-2k$.

So

$$-2k = 5$$
$$k = -\frac{5}{2}$$

b The remainder when $f(x)$ is divided by $x + 2$ is $f(-2)$ by the remainder theorem.

$$\begin{aligned} f(-2) &= 2(-2)^3 - 7(-2)^2 + (-2) + 10 \\ &= -16 - 28 - 2 + 10 \\ &= -36 \end{aligned}$$

9 The remainder when $P(x)$ is divided by $x - 1$ is $P(1)$ by the remainder theorem.

$$\begin{aligned} P(1) &= 2(1) + 4(1)^2 + 6(1)^3 + \dots + 100(1)^{50} \\ &= 2 + 4 + 6 + \dots + 100 \end{aligned}$$

Arithmetic progression with

$a = 2$
$d = 2$
$n = 50$

$P(1) =$ sum to 50 terms of the arithmetic progression

$= \frac{50}{2}\{2 \times 2 + (50 - 1) \times 2\}$
$= 25(4 + 98)$
$= 25 \times 102$
$= 2550$

Remember that the sum to n terms of an arithmetic progression with first term a and common difference d is $\frac{n}{2}\{2a + (n - 1)d\}$.

10 When $\mathrm{P}(x)$ is divided by $x + 1$ the remainder is -2.

Hence, by the remainder theorem, $\mathrm{P}(-1) = -2$.

$$\begin{aligned} \mathrm{P}(-1) &= -2 \\ 0 + 0 + 0 + c &= -2 \\ c &= -2 \end{aligned}$$

When $\mathrm{P}(x)$ is divided by $x + 2$ the remainder is 2.

Hence, by the remainder theorem, $\mathrm{P}(-2) = 2$.

$$\begin{aligned} \mathrm{P}(-2) &= 2 \\ 0 + 0 - b + c &= 2 \end{aligned}$$

Substituting $c = -2$:

$$\begin{aligned} -b - 2 &= 2 \\ b &= -4 \end{aligned}$$

When $\mathrm{P}(x)$ is divided by $x + 3$ the remainder is 10.

Hence, by the remainder theorem, $\mathrm{P}(-3) = 10$.

$$\begin{aligned} \mathrm{P}(-3) &= 10 \\ 0 + a(-2)(-1) + b(-2) + c &= 10 \end{aligned}$$

Substituting $c = -2$ and $b = -4$:

$$\begin{aligned} 2a - 2 \times (-4) - 2 &= 10 \\ 2a &= 12 - 8 = 4 \\ a &= 2 \end{aligned}$$

END-OF-CHAPTER REVIEW EXERCISE 1

1

$$|2x-3| = |5x+1|$$
$$(2x-3)^2 = (5x+1)^2$$
$$4x^2 - 12x + 9 = 25x^2 + 10x + 1$$
$$21x^2 + 22x - 8 = 0$$
$$(7x-2)(3x+4) = 0$$
$$x = \frac{2}{7},\ x = -\frac{4}{3}$$

Check:

$$\left|2\left(\frac{2}{7}\right) - 3\right| = \left|-\frac{17}{7}\right| = \frac{17}{7}, \quad \left|5\left(\frac{2}{7}\right) + 1\right| = \left|\frac{17}{7}\right| = \frac{17}{7}\ ✓$$
$$\left|2\left(-\frac{4}{3}\right) - 3\right| = \left|-\frac{17}{3}\right| = \frac{17}{3}, \quad \left|5\left(-\frac{4}{3}\right) + 1\right| = \left|-\frac{17}{3}\right| = \frac{17}{3}\ ✓$$

Solutions are:

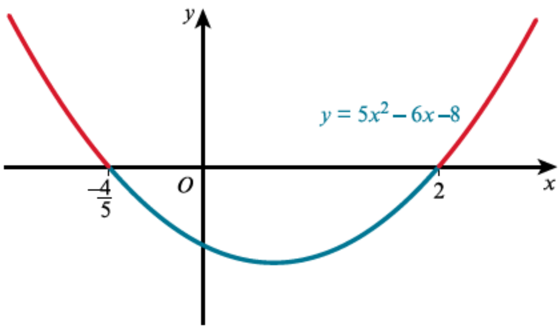
$$x = \frac{2}{7} \text{ or } x = -\frac{4}{3}$$

2

$$|5x-3| \geqslant 7$$
$$(5x-3)^2 \geqslant 49$$
$$25x^2 - 30x + 9 \geqslant 49$$
$$25x^2 - 30x - 40 \geqslant 0$$
$$5x^2 - 6x - 8 \geqslant 0$$
$$(5x+4)(x-2) \geqslant 0$$

Critical values are:

$$x = -\frac{4}{5},\ x = 2$$



Hence $x \leqslant -\frac{4}{5}$ or $x \geqslant 2$

3 Sketch the graphs of $y = |2x-3|$ and $y = |2-x|$.

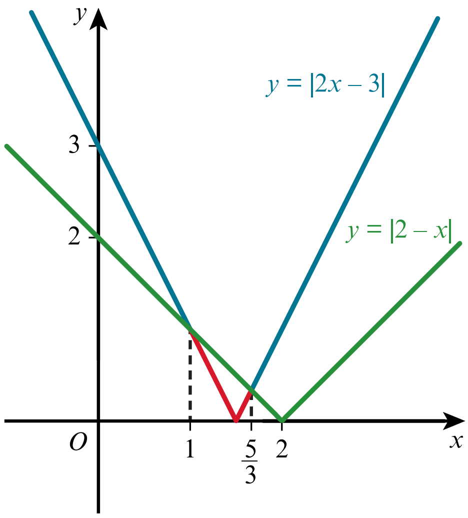
Graphs intersect where:

$$|2x-3| = |2-x|$$
$$(2x-3)^2 = (2-x)^2$$
$$4x^2 - 12x + 9 = 4 - 4x + x^2$$
$$3x^2 - 8x + 5 = 0$$
$$(3x-5)(x-1) = 0$$

Critical values are:

$$x = \frac{5}{3},\ x = 1$$

> Even if you feel that you could write down the final solution to an inequality without drawing any kind of diagram, you are not advised to do this. Not only would there be an increased risk to accuracy, but every mathematical argument should be complete. The diagram is part of the argument.



The graph of $y = |2x - 3|$ lies below the graph of $y = |2 - x|$ when:

$1 < x < \frac{5}{3}$

4 $|x^2 - 14| = 11$

$x^2 - 14 = 11$ or $x^2 - 14 = -11$

$x^2 = 25$ $\quad x^2 = 3$

$x = \pm 5$ $\quad x = \pm\sqrt{3}$

Check:

$|5^2 - 14| = |25 - 14| = 11$ ✓

$|(-5)^2 - 14| = |25 - 14| = 11$ ✓

$|(\sqrt{3})^2 - 14| = |3 - 14| = |-11| = 11$ ✓

$|(-\sqrt{3})^2 - 14| = |3 - 14| = |-11| = 11$ ✓

Solutions are:

$x = \pm 5$ or $x = \pm\sqrt{3}$

5 **a** $x - 4$ is a factor of $p(x)$, so $p(4) = 0$ by the factor theorem.

$$a(4)^3 - 13(4)^2 - 41(4) - 2a = 0$$
$$64a - 208 - 164 - 2a = 0$$
$$62a = 372$$
$$a = \frac{372}{62} = 6$$

b Substituting $a = 6$ into $p(x)$:

$p(x) = 6x^3 - 13x^2 - 41x - 12$

$x - 4$ is a factor of $p(x)$, so using long division to divide $p(x)$ by $x - 4$:

$$\begin{array}{r} 6x^2 + 11x + 3 \\ x - 4 \overline{) 6x^3 - 13x^2 - 41x - 12} \\ \underline{6x^3 - 24x^2} \qquad\qquad \\ 11x^2 - 41x \qquad \\ \underline{11x^2 - 44x} \qquad \\ 3x - 12 \\ \underline{3x - 12} \\ 0 \end{array}$$

$$6x^3 - 13x^2 - 41x - 12 = (x - 4)(6x^2 + 11x + 3)$$
$$= (x - 4)(3x + 1)(2x + 3)$$

6 a $\mathrm{f}(x)=6x^3-23x^2-38x+15$

$$\begin{aligned}\mathrm{f}(5)&=6(5)^3-23(5)^2-38(5)+15\\&=6\times125-23\times25-38\times5+15\\&=0\end{aligned}$$

Hence $x-5$ is a factor of $\mathrm{f}(x)$ by the factor theorem.

Using long division to divide $\mathrm{f}(x)$ by $x-5$:

$$\begin{array}{r}6x^2+7x-3\\ x-5\,\big)\,\overline{6x^3-23x^2-38x+15}\\ \underline{6x^3-30x^2}\qquad\qquad\\ 7x^2-38x\qquad\\ \underline{7x^2-35x}\qquad\\ -3x+15\\ \underline{-3x+15}\\ 0\end{array}$$

$$\begin{aligned}6x^3-23x^2-38x+15&=(x-5)(6x^2+7x-3)\\&=(x-5)(3x-1)(2x+3)\end{aligned}$$

b $\mathrm{f}(|x|)$ simply means the function $\mathrm{f}(x)$ but with x replaced by $|x|$ throughout.

$\mathrm{f}(|x|)=0$

So

$(|x|-5)(3|x|-1)(2|x|+3)=0$

$|x|=5,\ \dfrac{1}{3}$ or $-\dfrac{3}{2}$

$x=\pm5,\pm\dfrac{1}{3}$

There are no further solutions because it is not possible for $|x|$ to be equal to a negative number.

7 $\mathrm{f}(x)=x^3-5x^2+ax+b$

$x+2$ is a factor of $\mathrm{f}(x)$, so $\mathrm{f}(-2)=0$ by the factor theorem.

$$\begin{aligned}\mathrm{f}(-2)&=0\\(-2)^3-5(-2)^2+a(-2)+b&=0\\-8-20-2a+b&=0\\2a-b&=-28\ \ldots\ldots\ [1]\end{aligned}$$

When $\mathrm{f}(x)$ is divided by $x-1$ the remainder is -6.

Hence, by the remainder theorem, $\mathrm{f}(1)=-6$.

$$\begin{aligned}\mathrm{f}(1)&=-6\\(1)^3-5(1)^2+a(1)+b&=-6\\1-5+a+b&=-6\\a+b&=-2\ \ldots\ldots\ldots\ldots\ [2]\end{aligned}$$

[1] + [2]:

$3a=-30$

$a=-10$

Substituting $a=-10$ into [2]:

$-10+b=-2$

$b=8$

8 a

$$\begin{array}{r}x-3\\ x^2-2x-1\,\big)\,\overline{x^3-5x^2+7x-3}\\ \underline{x^3-2x^2-x}\qquad\\ -3x^2+8x-3\\ \underline{-3x^2+6x+3}\\ 2x-6\end{array}$$

Quotient $= x - 3$

Remainder $= 2x - 6$

b $p(x) = x^3 - 5x^2 + 7x - 3$

$p(3) = (3)^3 - 5(3)^2 + 7(3) - 3$

$= 27 - 45 + 21 - 3 = 0$

$p(3) = 0$, so, by the factor theorem, $x - 3$ is a factor of $p(x)$.

9 **a** $p(x) = 4x^4 + 4x^3 - 7x^2 - 4x + 8$

$$\begin{array}{r} 4x^2 + 4x - 3 \\ x^2 + 0x - 1 \overline{\smash{)}4x^4 + 4x^3 - 7x^2 - 4x + 8} \\ \underline{4x^4 + 0x^3 - 4x^2} \qquad\qquad \\ 4x^3 - 3x^2 - 4x \qquad \\ \underline{4x^3 + 0x^2 - 4x} \qquad \\ -3x^2 + 0x + 8 \\ \underline{-3x^2 + 0x + 3} \\ 5 \end{array}$$

Quotient $= 4x^2 + 4x - 3$

Remainder $= 5$

b $(x^2 - 1)(4x^2 + 4x - 3) + 5 = 4x^4 + 4x^3 - 7x^2 - 4x + 8$

$4x^4 + 4x^3 - 7x^2 - 4x + 3 = (x^2 - 1)(4x^2 + 4x - 3)$

$= (x - 1)(x + 1)(2x + 3)(2x - 1)$

$(x - 1)(x + 1)(2x + 3)(2x - 1) = 0$

Solutions are:

$x = 1, -1, -\frac{3}{2}, \frac{1}{2}$

10 **a** $f(x) = x^4 - 48x^2 - 21x - 2$

$= (x^2 + kx + 2)(x^2 - kx - 1)$

$= x^4 - kx^3 - x^2 + kx^3 - k^2x^2 - kx + 2x^2 - 2kx - 2$

$= x^4 + (-k + k)x^3 + (-1 - k^2 + 2)x^2 + (-k - 2k)x - 2$

Comparing the coefficients of x, which must match:

$-3k = -21$

$k = 7$

b Substituting the value for k from part **a**:

$x^4 - 48x^2 - 21x - 2 = (x^2 + 7x + 2)(x^2 - 7x - 1) = 0$

$x^2 + 7x + 2 = 0$ or $x^2 - 7x - 1 = 0$

$x = \frac{-7 \pm \sqrt{7^2 - 4 \times 1 \times 2}}{2 \times 1}$ $\quad x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \times 1 \times (-1)}}{2 \times 1}$

$= \frac{-7 \pm \sqrt{41}}{2}$ $\quad = \frac{7 \pm \sqrt{53}}{2}$

Avoid reaching for your calculator unless specifically asked to give a decimal. Exact solutions are appropriate for algebraic equations.

11 **a** $p(x) = 2x^4 + 3x^3 - 12x^2 - 7x + a$

$2x - 1$ is a factor of $p(x)$, so $p\left(\frac{1}{2}\right) = 0$ by the factor theorem.

$2\left(\frac{1}{2}\right)^4 + 3\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 - 7\left(\frac{1}{2}\right) + a = 0$

$\frac{2}{16} + \frac{3}{8} - \frac{12}{4} - \frac{7}{2} + a = 0$

$$a = -\frac{2 + 6 - 48 - 56}{16} = \frac{96}{16} = 6$$

b Substituting $a = 6$ into $p(x)$:

$$\begin{aligned} p(x) &= 2x^4 + 3x^3 - 12x^2 - 7x + 6 \\ p(-3) &= 2(-3)^4 + 3(-3)^3 - 12(-3)^2 - 7(-3) + 6 \\ &= 162 - 81 - 108 + 21 + 6 = 0 \end{aligned}$$

$p(-3) = 0$, so, by the factor theorem, $x + 3$ is also a factor of $p(x)$.

$x + 3$ and $2x - 1$ are factors of $p(x)$, so $(x + 3)(2x - 1)$ is a factor of $p(x)$.

Using long division to divide $p(x)$ by $(x + 3)(2x - 1) = 2x^2 + 5x - 3$:

$$\begin{array}{r} x^2 - x - 2 \\ 2x^2 + 5x - 3\,\big)\overline{2x^4 + 3x^3 - 12x^2 - 7x + 6} \\ \underline{2x^4 + 5x^3 - 3x^2} \qquad\qquad \\ -2x^3 - 9x^2 - 7x \qquad \\ \underline{-2x^3 - 5x^2 + 3x} \qquad \\ -4x^2 - 10x + 6 \\ \underline{-4x^2 - 10x + 6} \\ 0 \end{array}$$

$$\begin{aligned} 2x^4 + 3x^3 - 12x^2 - 7x + 6 &= (2x^2 + 5x - 3)(x^2 - x - 2) \\ &= (2x - 1)(x + 3)(x - 2)(x + 1) \end{aligned}$$

Remember that if any two expressions are factors of another, larger, expression then so is their product.

12 **a** $p(x) = 3x^3 + ax^2 - 36x + 20$

$x - 2$ is a factor of $p(x)$, so $p(2) = 0$ by the factor theorem.

$$\begin{aligned} 3(2)^3 + a(2)^2 - 36(2) + 20 &= 0 \\ 24 + 4a - 72 + 20 &= 0 \\ 4a &= 28 \\ a &= 7 \end{aligned}$$

b Substituting the value for a from part **a**:

$p(x) = 3x^3 + 7x^2 - 36x + 20$

Using long division to divide $p(x)$ by $x - 2$:

$$\begin{array}{r} 3x^2 + 13x - 10 \\ x - 2\,\big)\overline{3x^3 + 7x^2 - 36x + 20} \\ \underline{3x^3 - 6x^2} \qquad\qquad\quad \\ 13x^2 - 36x \qquad \\ \underline{13x^2 - 26x} \qquad \\ -10x + 20 \\ \underline{-10x + 20} \\ 0 \end{array}$$

$$\begin{aligned} 3x^3 + 7x^2 - 36x + 20 &= (x - 2)(3x^2 + 13x - 10) \\ &= (x - 2)(3x - 2)(x + 5) \end{aligned}$$

$(x - 2)(3x - 2)(x + 5) = 0$

$x = 2, \frac{2}{3}, -5$

13 **a** The remainder when $f(x)$ is divided by $x - 2$ is $f(2)$ by the remainder theorem.

$$\begin{aligned} f(2) &= 2(2)^3 + 5(2)^2 - 7(2) + 11 \\ &= 16 + 20 - 14 + 11 \\ &= 33 \end{aligned}$$

The remainder is 33.

b Using long division to divide $f(x)$ by $x^2 - 4x + 2$:

$$\begin{array}{r} 2x+13 \\ x^2-4x+2\overline{)2x^3+5x^2-7x+11} \\ \underline{2x^3-8x^2+4x} \\ 13x^2-11x+11 \\ \underline{13x^2-52x+26} \\ 41x-15 \end{array}$$

Quotient $= 2x + 13$

Remainder $= 41x - 15$

14 **a** $x-3$ and $x+1$ are both factors of $\mathrm{p}(x)$, so $\mathrm{p}(3)=\mathrm{p}(-1)=0$ by the factor theorem.

$$a(3)^3+b(3)^2-(3)+12=0$$
$$27a+9b=-9$$
$$3a+b=-1 \ldots\ldots\ldots\ldots\ldots [1]$$
$$a(-1)^3+b(-1)^2-(-1)+12=0$$
$$-a+b+1+12=0$$
$$a-b=13 \ldots\ldots\ldots [2]$$

[1] + [2]:

$$4a=12$$
$$a=3$$

Substituting $a=3$ into [2]:

$$3-b=13$$
$$b=-10$$

b $x-3$ and $x+1$ are both factors of $\mathrm{p}(x)$, so $(x-3)(x+1)=x^2-2x-3$ is also a factor of $\mathrm{p}(x)$.

Substituting the values for a and b from part **a** and using long division to divide $\mathrm{p}(x)$ by x^2-2x-3:

$$\begin{array}{r} 3x-4 \\ x^2-2x-3\overline{)3x^3-10x^2-x+12} \\ \underline{3x^3-6x^2-9x} \\ -4x^2+8x+12 \\ \underline{-4x^2+8x+12} \\ 0 \end{array}$$

$$\begin{aligned} 3x^3-10x^2-x+12 &= (x^2-2x-3)(3x-4) \\ &= (x-3)(x+1)(3x-4) \end{aligned}$$

$3x-4$ is the remaining linear factor of $\mathrm{p}(x)$.

15 **a** When $\mathrm{P}(x)$ is divided by $x+2$ the remainder is -12.

Hence, by the remainder theorem, $\mathrm{P}(-2)=-12$.

$$\begin{aligned} \mathrm{P}(-2) &= -12 \\ 6(-2)^3+(-2)^2+a(-2)-10 &= -12 \\ -48+4-2a-10 &= -12 \\ 2a &= -42 \\ a &= -21 \end{aligned}$$

$$\begin{aligned} \mathrm{P}\left(-\frac{1}{2}\right) &= 6\left(-\frac{1}{2}\right)^3+\left(-\frac{1}{2}\right)^2-21\left(-\frac{1}{2}\right)-10 \\ &= -\frac{6}{8}+\frac{1}{4}+\frac{21}{2}-10=\frac{-6+2+84-80}{8}=0 \end{aligned}$$

$\mathrm{P}\left(-\frac{1}{2}\right)=0$, so $2x+1$ is a factor of $\mathrm{P}(x)$ by the factor theorem.

b Substituting the value for a from part **a** and using long division to divide $\mathrm{P}(x)$ by $2x+1$:

$$\begin{array}{r} 3x^2-x-10 \\ 2x+1\overline{)6x^3+x^2-21x-10} \end{array}$$

$$\begin{array}{r} \underline{6x^3 + 3x^2} \\ -2x^2 - 21x \\ \underline{-2x^2 - x} \\ -20x - 10 \\ \underline{-20x - 10} \\ 0 \end{array}$$

$$6x^3 + x^2 - 21x - 10 = 0$$
$$(2x+1)(3x^2 - x - 10) = 0$$
$$(2x+1)(3x+5)(x-2) = 0$$
$$x = -\frac{1}{2}, -\frac{5}{3}, 2$$

16 **a** $x+2$ and $x-3$ are factors of $\mathrm{p}(x)$, so $\mathrm{p}(-2) = \mathrm{p}(3) = 0$ by the factor theorem.

$$2(-2)^3 + a(-2)^2 + b(-2) + 6 = 0$$
$$-16 + 4a - 2b + 6 = 0$$
$$4a - 2b = 10$$
$$2a - b = 5 \text{ .... [1]}$$

$$2(3)^3 + a(3)^2 + b(3) + 6 = 0$$
$$54 + 9a + 3b + 6 = 0$$
$$9a + 3b = -60$$
$$3a + b = -20 \text{ .... [2]}$$

[1] + [2]:

$$5a = -15$$
$$a = -3$$

Substituting $a = -3$ into [2]:

$$-9 + b = -20$$
$$b = -11$$

b Substituting the values for a and b from part **a** and using long division to divide $\mathrm{p}(x)$ by $(x+2)(x-3) = x^2 - x - 6$:

$$\begin{array}{r} 2x - 1 \\ x^2 - x - 6 \overline{)2x^3 - 3x^2 - 11x + 6} \\ \underline{2x^3 - 2x^2 - 12x} \\ -x^2 + x + 6 \\ \underline{-x^2 + x + 6} \\ 0 \end{array}$$

$$2x^3 - 3x^2 - 11x + 6 = (x^2 - x - 6)(2x - 1)$$
$$= (x-3)(x+2)(2x-1)$$

17 **a** $x-2$ is a factor of $\mathrm{P}(x)$, so $\mathrm{P}(2) = 0$ by the factor theorem.

$$(2)^3 + a(2)^2 + b = 0$$

$$4a + b = -8 \text{ .... [1]}$$

When $\mathrm{Q}(x)$ is divided by $x+1$ the remainder is -15.

Hence, by the remainder theorem, $\mathrm{Q}(-1) = -15$.

$$(-1)^3 + b(-1)^2 + a = -15$$

$$a + b = -14 \text{ .... [2]}$$

[1] − [2]:

$$3a = 6$$
$$a = 2$$

Substituting $a = 2$ into [2]:

$$2 + b = -14$$
$$b = -16$$

b Substituting the values for a and b from part **a**:

$$
\begin{aligned}
P(x) - Q(x) &= x^3 + 2x^2 - 16 - (x^3 - 16x^2 + 2) \\
&= x^3 + 2x^2 - 16 - x^3 + 16x^2 - 2 \\
&= 18x^2 - 18
\end{aligned}
$$

Minimum value when $x = 0$

Minimum $= -18$

> Remember that, provided x is a real number, any squared expression in x has a minimum value of zero.

18 **a** Using long division to divide $p(x)$ by $x - 1$:

$$
\begin{array}{r}
5x^2 - 8x + 9 \\
x - 1 \overline{)\,5x^3 - 13x^2 + 17x - 7} \\
\underline{5x^3 - 5x^2} \qquad\qquad\quad \\
-8x^2 + 17x \qquad \\
\underline{-8x^2 + 8x} \qquad \\
9x - 7 \\
\underline{9x - 9} \\
2
\end{array}
$$

Quotient $= 5x^2 - 8x + 9$

Remainder $= 2$

b Using the result from part **a**:

$5x^3 - 13x^2 + 17x - 7 = (x - 1)(5x^2 - 8x + 9) + 2$

So

$$
\begin{aligned}
5x^3 - 13x^2 + 17x - 9 &= (x - 1)(5x^2 - 8x + 9) \\
5x^3 - 13x^2 + 17x - 9 &= 0
\end{aligned}
$$

So

$(x - 1)(5x^2 - 8x + 9) = 0$

$$
\begin{aligned}
x = 1 \quad \text{or} \quad & 5x^2 - 8x + 9 = 0 \\
& b^2 - 4ac = (-8)^2 - 4 \times 5 \times 9 \\
& \qquad\quad = 64 - 180 < 0
\end{aligned}
$$

The quadratic, therefore, has no real roots. The cubic equation has only one real root, $x = 1$.

19 **a** $x + 2$ is a factor of $f(x)$, so, by the factor theorem, $f(-2) = 0$.

$$
\begin{aligned}
4(-2)^3 + k(-2)^2 - 65(-2) + 18 &= 0 \\
4k - 32 + 130 + 18 &= 0 \\
4k &= -116 \\
k &= -29
\end{aligned}
$$

b Substituting the value for k from part **a** and using long division to divide $f(x)$ by $x + 2$:

$$
\begin{array}{r}
4x^2 - 37x + 9 \\
x + 2 \overline{)\,4x^3 - 29x^2 - 65x + 18} \\
\underline{4x^3 + 8x^2} \qquad\qquad\qquad \\
-37x^2 - 65x \qquad \\
\underline{-37x^2 - 74x} \qquad \\
9x + 18 \\
\underline{9x + 18} \\
0
\end{array}
$$

$$
\begin{aligned}
4x^3 - 29x^2 - 65x + 18 &= (x + 2)(4x^2 - 37x + 9) \\
&= (x + 2)(4x - 1)(x - 9)
\end{aligned}
$$

$$
\begin{aligned}
(x + 2)(4x - 1)(x - 9) &= 0 \\
x &= -2,\ \tfrac{1}{4},\ 9
\end{aligned}
$$

c $x^2 = -2$ No real solutions.

$x^2 = 9 \qquad x = \pm\sqrt{9} = \pm 3$

$x^2 = \frac{1}{4} \qquad x = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

The roots of $f(x^2) = 0$ are:

$x = \pm 3, \pm\frac{1}{2}$

20 **a** When $f(x)$ is divided by $x + 2$ the remainder is 8.

Hence, by the remainder theorem, $f(-2) = 8$.

$2(-2)^3 - 5(-2)^2 + a(-2) + b = 8$

$-16 - 20 - 2a + b = 8$

$2a - b = -44$ .... [1]

When $f(x)$ is divided by $x - 1$ the remainder is 50.

Hence, by the remainder theorem, $f(1) = 50$.

$2(1)^3 - 5(1)^2 + a(1) + b = 50$

$2 - 5 + a + b = 50$

$a + b = 53$ .... [2]

[1] + [2]:

$3a = 9$

$a = 3$

Substituting $a = 3$ into [2]:

$3 + b = 53$

$b = 50$

b Substituting the values for a and b from part **a** and using long division to divide $f(x)$ by $x^2 - x + 2$:

$$\begin{array}{r} 2x - 3 \\ x^2 - x + 2\overline{)2x^3 - 5x^2 + 3x + 50} \\ \underline{2x^3 - 2x^2 + 4x} \qquad \\ -3x^2 - x + 50 \\ \underline{-3x^2 + 3x - 6} \\ -4x + 56 \end{array}$$

Quotient $= 2x - 3$

Remainder $= -4x + 56$

21 **a** $x + 2$ is a factor of $f(x)$, so, by the factor theorem, $f(-2) = 0$.

$2(-2)^3 - 9(-2)^2 + a(-2) + b = 0$

$-16 - 36 - 2a + b = 0$

$2a - b = -52$ .... [1]

When $f(x)$ is divided by $x + 1$ the remainder is 30.

Hence, by the remainder theorem, $f(-1) = 30$.

$2(-1)^3 - 9(-1)^2 + a(-1) + b = 30$

$-2 - 9 - a + b = 30$

$a - b = -41$ .... [2]

[1] – [2]:

$a = -11$

Substituting $a = -11$ into [2]:

$-11 - b = -41$

$b = 30$

b Substituting the values for a and b from part **a** and using long division to divide $f(x)$ by $x + 2$:

$$\begin{array}{r} 2x^2 - 13x + 15 \\ x + 2\overline{)2x^3 - 9x^2 - 11x + 30} \end{array}$$

$$\underline{2x^3 + 4x^2}$$
$$- 13x^2 - 11x$$
$$\underline{- 13x^2 - 26x}$$
$$15x + 30$$
$$\underline{15x + 30}$$
$$0$$

$$\begin{aligned} 2x^3 - 9x^2 - 11x + 30 &= (x+2)(2x^2 - 13x + 15) \\ &= (x+2)(2x-3)(x-5) \end{aligned}$$

$(x+2)(2x-3)(x-5) = 0$

$x = -2, \frac{3}{2}, 5$

22 **i** Using long division to divide $\mathrm{f}(x)$ by $x^2 + x - 1$:

$$\begin{array}{r} x + 2 \\ x^2 + x - 1 \overline{\smash{)}\,x^3 + 3x^2 + 4x + 2} \\ \underline{x^3 + x^2 - x} \\ 2x^2 + 5x + 2 \\ \underline{2x^2 + 2x - 2} \\ 3x + 4 \end{array}$$

Quotient $= x + 2$

Remainder $= 3x + 4$

ii $\mathrm{f}(x) = x^3 + 3x^2 + 4x + 2$

$$\begin{aligned} \mathrm{f}(-1) &= (-1)^3 + 3(-1)^2 + 4(-1) + 2 \\ &= -1 + 3 - 4 + 2 = 0 \end{aligned}$$

$\mathrm{f}(-1) = 0$, so $x + 1$ is a factor of $\mathrm{f}(x)$ by the factor theorem.

23 **i** When $\mathrm{p}(x)$ is divided by $2x - 1$ the remainder is 10.

Hence, by the remainder theorem, $\mathrm{p}\left(\frac{1}{2}\right) = 10$.

$$\begin{aligned} 4\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 + 9\left(\frac{1}{2}\right) + 9 &= 10 \\ \frac{4}{8} + \frac{a}{4} + \frac{9}{2} + 9 &= 10 \\ \frac{4 + 2a + 36}{8} &= 1 \\ 2a + 40 &= 8 \\ 2a &= -32 \\ a &= -16 \end{aligned}$$

Substituting $a = -16$ into $\mathrm{p}(x)$:

$$\begin{aligned} \mathrm{p}(x) &= 4x^3 - 16x^2 + 9x + 9 \\ \mathrm{p}(3) &= 4(3)^3 - 16(3)^2 + 9(3) + 9 \\ &= 108 - 144 + 27 + 9 = 0 \end{aligned}$$

$\mathrm{p}\,(3) = 0$, so $x - 3$ is a factor of $\mathrm{p}(x)$ by the factor theorem.

ii Substituting the value for a from part **i** and using long division to divide $\mathrm{p}(x)$ by $x - 3$:

$$\begin{array}{r} 4x^2 - 4x - 3 \\ x - 3 \overline{\smash{)}\,4x^3 - 16x^2 + 9x + 9} \\ \underline{4x^3 - 12x^2} \\ - 4x^2 + 9x \\ \underline{- 4x^2 + 12x} \\ - 3x + 9 \\ \underline{- 3x + 9} \\ 0 \end{array}$$

$$4x^3 - 16x^2 + 9x + 9 = (x-3)(4x^2 - 4x - 3)$$
$$= (x-3)(2x+1)(2x-3)$$
$$(x-3)(2x+1)(2x-3) = 0$$
$$x = 3, -\frac{1}{2}, \frac{3}{2}$$

24 **i** $2x + 3$ is a factor of $\mathrm{p}(x)$, so, by the factor theorem, $\mathrm{p}\left(-\frac{3}{2}\right) = 0$.

$$a\left(-\frac{3}{2}\right)^3 - 5\left(-\frac{3}{2}\right)^2 + b\left(-\frac{3}{2}\right) + 9 = 0$$
$$-\frac{27}{8}a - \frac{45}{4} - \frac{3}{2}b + 9 = 0$$
$$-27a - 90 - 12b = -72$$
$$27a + 12b = -18$$
$$9a + 4b = -6 \text{ .... [1]}$$

When $\mathrm{p}(x)$ is divided by $x + 1$ the remainder is 8.

Hence, by the remainder theorem, $\mathrm{p}(-1) = 8$.

$$a(-1)^3 - 5(-1)^2 + b(-1) + 9 = 8$$
$$-a - 5 - b + 9 = 8$$
$$a + b = -4 \text{ .... [2]}$$

$4 \times$ [2]:

$$4a + 4b = -16 \text{ .... [3]}$$

[1] − [3]:

$$5a = 10$$
$$a = 2$$

Substituting $a = 2$ into [2]:

$$2 + b = -4$$
$$b = -6$$

ii Substituting the values for a and b from part **i** and using long division to divide $\mathrm{p}(x)$ by $2x + 3$:

$$\begin{array}{r} x^2 - 4x + 3 \\ 2x+3\overline{\smash{)}\,2x^3 - 5x^2 - 6x + 9} \\ \underline{2x^3 + 3x^2} \qquad\qquad \\ -8x^2 - 6x \quad\; \\ \underline{-8x^2 - 12x} \quad\; \\ 6x + 9 \\ \underline{6x + 9} \\ 0 \end{array}$$

$$2x^3 - 5x^2 - 6x + 9 = (2x+3)(x^2 - 4x + 3)$$
$$= (2x+3)(x-3)(x-1)$$

Chapter 2
Logarithmic and exponential functions

EXERCISE 2A

You can think of each logarithm as a 'question'. $a = \log_b c$ is the same as saying that a is the answer to the question 'What power of b gives the answer c?'.

1 **a** **Method 1**

$$\begin{aligned} \text{base} &= 10 \\ \text{index} &= 2 \\ 2 &= \log_{10} 100 \end{aligned}$$

Method 2

In the second method shown here, first you take logs to base 10 of both sides and then you use the fact that $\log_{10} 10^x = x$. You can try this alternative method in the other parts of Question 1.

$$\begin{aligned} 10^2 &= 100 \\ \log_{10} 10^2 &= \log_{10} 100 \\ 2 &= \log_{10} 100 \end{aligned}$$

b

$$\begin{aligned} \text{base} &= 10 \\ \text{index} &= x \\ x &= \log_{10} 200 \end{aligned}$$

c

$$\begin{aligned} \text{base} &= 10 \\ \text{index} &= x \\ x &= \log_{10} 0.05 \end{aligned}$$

2 **a**

$$\begin{aligned} \text{base} &= 10 \\ \text{index} &= x \\ x &= \log_{10} 52 = 1.72 \end{aligned}$$

b

$$\begin{aligned} \text{base} &= 10 \\ \text{index} &= x \\ x &= \log_{10} 250 = 2.40 \end{aligned}$$

c

$$\begin{aligned} \text{base} &= 10 \\ \text{index} &= x \\ x &= \log_{10} 0.48 = -0.319 \end{aligned}$$

3 **a** **Method 1**

$$\begin{aligned} \text{base} &= 10 \\ \text{index} &= 4 \\ 10\,000 &= 10^4 \end{aligned}$$

Method 2

In the second method shown here, first you take 10 to the power of both sides and then you use the fact that $10^{\log_{10} x} = x$. You can try this alternative method in the other parts of Question 3.

$$\begin{aligned} \log_{10} 10\,000 &= 4 \\ 10^{\log_{10} 10\,000} &= 10^4 \\ 10\,000 &= 10^4 \end{aligned}$$

b $\text{base} = 10$

$\text{index} = 1.2$

$x = 10^{1.2}$

c $\text{base} = 10$

$\text{index} = -0.6$

$x = 10^{-0.6}$

4 **a** $\text{base} = 10$

$\text{index} = 1.88$

$x = 10^{1.88} = 75.9$

b $\text{base} = 10$

$\text{index} = 2.76$

$x = 10^{2.76} = 575$

c $\text{base} = 10$

$\text{index} = -1.4$

$x = 10^{-1.4} = 0.0398$

5 **c** $\log_{10}\left(10\sqrt{10}\right)$

$= \log_{10}\left(10^1 \times 10^{\frac{1}{2}}\right)$

$= \log_{10} 10^{\frac{3}{2}}$

$= \frac{3}{2}$

f $\log_{10}\left(\frac{100}{\sqrt{1000}}\right)$

$= \log_{10}\left(\frac{100}{\sqrt{10^3}}\right)$

$= \log_{10}\left(\frac{10^2}{10^{\frac{3}{2}}}\right)$

$= \log_{10}\left(10^{2-\frac{3}{2}}\right)$

$= \log_{10} 10^{\frac{1}{2}}$

$= \frac{1}{2}$

Note that this question relies on the laws of indices. In worked solution **5 f** the denominator of the fraction was simplified using the fact that $\sqrt{x^n} = (x^n)^{\frac{1}{2}} = x^{\frac{n}{2}}$.

6 $\text{f}(x) = 10^x - 3$

For simplicity, writing y instead of $\text{f}(x)$:

$y = 10^x - 3$

$10^x = y + 3$

$\text{base} = 10$

$\text{index} = x$

$x = \log_{10}(y + 3)$

So the inverse function is:

$\text{f}^{-1}(x) = \log_{10}(x + 3)$

When finding the inverse of a function, you need to swap the inputs and the outputs. This is why x and y were swapped in worked solution **6**.

7 Let $x = \log_{10} p$ and $y = \log_{10} q$.

$4x^2 + 2y^2 = 9$

Both x^2 and y^2 are always positive.

p will be at its greatest when x is at its greatest.

This happens when y is as small as possible.

When $y = 0$:

$$\begin{aligned} 4x^2 &= 9 \\ x^2 &= \frac{9}{4} \\ x &= \pm\sqrt{\frac{9}{4}} = \pm\frac{3}{2} \end{aligned}$$

So

$$\begin{aligned} \log_{10} p &= \pm\frac{3}{2} \\ p &= 10^{\frac{3}{2}} = 31.6 \end{aligned}$$

or $p = 10^{-\frac{3}{2}} = 0.0316$

So, the greatest value of p is $p = 10^{\frac{3}{2}} = 10^1 \times 10^{\frac{1}{2}} = 10\sqrt{10}$

EXERCISE 2B

1 **d** **Method 1**

$$2^{-10} = \frac{1}{1024}$$
$$\text{base} = 2$$
$$\text{index} = -10$$
$$-10 = \log_2\left(\frac{1}{1024}\right)$$

Method 2

You can take logs to base 2 of both sides, as shown in the second method here.

$$2^{-10} = \frac{1}{1024}$$
$$\log_2 2^{-10} = \log_2\left(\frac{1}{1024}\right)$$
$$-10 = \log_2\left(\frac{1}{1024}\right)$$

g

$$a^b = c$$
$$\text{base} = a$$
$$\text{index} = b$$
$$b = \log_a c$$

2 **b** **Method 1**

$$\log_3 81 = 4$$
$$\text{base} = 3$$
$$\text{index} = 4$$
$$3^4 = 81$$

Method 2

$$\log_3 81 = 4$$
$$3^{\log_3 81} = 3^4$$
$$81 = 3^4$$

c

$$\log_x 5 = y$$
$$\text{base} = x$$
$$\text{index} = y$$
$$x^y = 5$$

3 **a**

$$\log_2 x = 3$$
$$\text{base} = 2$$
$$\text{index} = 3$$
$$x = 2^3 = 8$$

b

$$\log_3 x = 2$$
$$\text{base} = 3$$
$$\text{index} = 2$$
$$x = 3^2 = 9$$

4 **a**

$$\log_3(x+5) = 2$$
$$\text{base} = 3$$
$$\text{index} = 2$$
$$x + 5 = 3^2$$
$$x + 5 = 9$$
$$x = 4$$

b $\log_2(3x-1)=5$

$$\begin{aligned}\text{base}&=2\\ \text{index}&=5\\ 3x-1&=2^5\\ 3x-1&=32\\ 3x&=33\\ x&=11\end{aligned}$$

c $\log_y(7-2x)=0$

$$\begin{aligned}\text{base}&=y\\ \text{index}&=0\\ 7-2x&=y^0\\ 7-2x&=1\\ 2x&=6\\ x&=3\end{aligned}$$

5 **d** $\log_2 0.125$

$=\log_2\left(\frac{1}{8}\right)$

$=\log_2\left(2^{-3}\right)$

$=-3$

Remember that the logarithm of any number less than 1 will always be negative whatever the base of the logarithm.

g $\log_3\left(\frac{\sqrt{3}}{3}\right)$

$=\log_3\left(3^{\frac{1}{2}-1}\right)$

$=\log_3 3^{-\frac{1}{2}}$

$=-\frac{1}{2}$

6 **c** $\log_x\left(x^2\sqrt{x}\right)$

$=\log_x\left(x^{2+\frac{1}{2}}\right)$

$=\log_x x^{\frac{5}{2}}$

$=\frac{5}{2}$

h $\log_x\left(\frac{x^2}{\sqrt{x}}\right)^3$

$=\log_x\left(x^{2-\frac{1}{2}}\right)^3$

$=\log_x\left(x^{\frac{3}{2}}\right)^3$

$=\log_x x^{\frac{9}{2}}$

$=\frac{9}{2}$

7 $f(x)=1+\log_2(x-3)$

For simplicity, writing y instead of $f(x)$:

$$\begin{aligned}y&=1+\log_2(x-3)\\ \log_2(x-3)&=y-1\\ \text{base}&=2\\ \text{index}&=y-1\\ x-3&=2^{y-1}\\ x&=3+2^{y-1}\end{aligned}$$

Swapping x and y:

$$y = 3 + 2^{x-1}$$
$$f^{-1}(x) = 3 + 2^{x-1}$$

When finding inverse functions, you need to swap the role of 'inputs' and 'outputs'. This is why x and y are swapped during the process of finding the inverse in worked solution **7**.

8 **a** $\log_2(\log_3 x) = -1$

base $= 2$

index $= -1$

$\log_3 x = 2^{-1}$

base $= 3$

index $= 2^{-1}$

$$x = 3^{(2^{-1})} = 3^{\frac{1}{2}} = \sqrt{3}$$

b $\log_4 2^{3-5x} = x^2$

base $= 4$

index $= x^2$

$$2^{3-5x} = 4^{x^2}$$
$$2^{3-5x} = \left(2^2\right)^{x^2}$$
$$2^{3-5x} = 2^{2x^2}$$

Equating the indices:

$$2x^2 = 3 - 5x$$
$$2x^2 + 5x - 3 = 0$$
$$(2x - 1)(x + 3) = 0$$
$$x = \frac{1}{2} \text{ or } x = -3$$

9 $\log_3 9 = \log_3 3^2 = 2$

$\log_2 8 = \log_2 2^3 = 3$

$\log_3 9 < \log_3 20 < \log_3 27$ so $2 < \log_3 20 < 3$

so $\log_3 9 < \log_3 20 < \log_2 8$

$\log_3 3 < \log_3 4 < \log_3 9$ so $1 < \log_3 4 < 2$

so $1 < \log_3 4 < \log_3 9 < \log_3 20 < \log_2 8$

$\log_2 2 = 1$

so $\log_2 2 < \log_3 4 < \log_3 9 < \log_3 20 < \log_2 8$

$\log_4 3 < \log_4 4$ so $\log_4 3 < 1$

so $\log_4 3 < \log_2 2 < \log_3 4 < \log_3 9 < \log_3 20 < \log_2 8$

$\log_2 2 < \log_2 3 < \log_2 4$ so $1 < \log_2 3 < 2$

So you now just need to compare $\log_2 3$ and $\log_3 4$, both of which lie between 1 and 2.

You can see the answer by looking at the sequence of integer powers of 2 and 3:

Sequence of powers of 2: $1, 2, 4, 8, 16 \ldots$

Sequence of powers of 3: $1, 3, 9, 27 \ldots$

Powers of 3 clearly increase much more quickly, so the power of 3 required to get 4 is going to be less than the power of 2 required to get 3.

So $\log_3 4 < \log_2 3$

The order is:

$\log_4 3 < \log_2 2 < \log_3 4 < \log_2 3 < \log_3 9 < \log_3 20 < \log_2 8$

EXERCISE 2C

1 **b** $\log_6 20 - \log_6 4$

$= \log_6 \left(\frac{20}{4}\right)$

$= \log_6 5$

e $1 + 2\log_2 3$

$= \log_2 2 + \log_2 3^2$

$= \log_2(2 \times 3^2)$

$= \log_2(2 \times 9)$

$= \log_2 18$

If you find that a question asks for a number to be added to a logarithm, as in part **e** of Question **1**, try to write the number in logarithmic form before continuing.

2 **a** $\log_2 40 - \log_2 5$

$= \log_2 \left(\frac{40}{5}\right)$

$= \log_2 8$

$= \log_2 2^3$

$= 3$

b $\log_6 20 + \log_6 5$

$= \log_6(20 \times 5)$

$= \log_6 100$

$= \log_6 10^2$

$= 2\log_6 10$

c $\log_2 60 - \log_2 5$

$= \log_2 \left(\frac{60}{5}\right)$

$= \log_2 12$

$= \log_2(3 \times 4)$

$= \log_2 3 + \log_2 4$

$= \log_2 3 + \log_2 2^2$

$= 2 + \log_2 3$

3 **a** $3\log_5 2 + \frac{1}{2}\log_5 36 - \log_5 12$

$= \log_5 2^3 + \log_5 36^{\frac{1}{2}} - \log_5 12$

$= \log_5 \left(\frac{2^3 \times \sqrt{36}}{12}\right)$

$= \log_5 \left(\frac{8 \times 6}{12}\right)$

$= \log_5 4$

$= \log_5 2^2$

$= 2\log_5 2$

Many questions, like these, rely on you spotting a number of powers or roots of numbers. The more of these you have committed to memory, the easier you will find it to complete these questions.

b $\frac{1}{2}\log_3 8 + \frac{1}{2}\log_3 18 - 1$

$= \frac{1}{2}\log_3 8 + \frac{1}{2}\log_3 18 - \log_3 3$

$= \frac{1}{2}\log_3 8 + \frac{1}{2}\log_3 18 - \frac{1}{2}\log_3 3^2$

$= \frac{1}{2}\log_3 \left(\frac{8 \times 18}{9}\right)$

$= \frac{1}{2}\log_3 16$

$= \log_3 16^{\frac{1}{2}}$

$= \log_3 \sqrt{16}$

$= \log_3 4$

$= \log_3 2^2$

$= 2\log_3 2$

4

$$8 = 2^3$$

$$0.25 = \frac{1}{4} = \frac{1}{2^2} = 2^{-2}$$

$$\frac{\log_5 8}{\log_5 0.25} = \frac{\log_5 2^3}{\log_5 2^{-2}}$$

$$= \frac{3\log_5 2}{-2\log_5 2}$$

$$= -\frac{3}{2}$$

5 **b** $\frac{\log_3 128}{\log_3 16}$

$= \frac{\log_3 2^7}{\log_3 2^4}$

$= \frac{7\log_3 2}{4\log_3 2}$

$= \frac{7}{4}$

c $\frac{\log_3 25}{\log_3 0.04} = \frac{\log_3 5^2}{\log_3 \left(\frac{1}{25}\right)}$

$= \frac{2\log_3 5}{\log_3 \left(5^{-2}\right)}$

$= \frac{2\log_3 5}{-2\log_3 5}$

$= -1$

6

$$\log_8 y = \log_8(x-2) - 2\log_8 x$$

$$\log_8 y = \log_8(x-2) - \log_8 x^2$$

$$\log_8 y = \log_8\left(\frac{x-2}{x^2}\right)$$

$$y = \frac{x-2}{x^2}$$

7 $\log_3(z-1) - \log_3 z = 1 + 3\log_3 y$

$$\log_3\left(\frac{z-1}{z}\right) = \log_3 3 + \log_3 y^3$$

$$\log_3\left(\frac{z-1}{z}\right) = \log_3\left(3y^3\right)$$

$$\frac{z-1}{z} = 3y^3$$

$$z - 1 = 3y^3 z$$

$$z - 3y^3 z = 1$$

$$z(1 - 3y^3) = 1$$

$$z = \frac{1}{1-3y^3}$$

8 **b** $\log_5(25x) = \log_5 25 + \log_5 x$

$= \log_5 5^2 + \log_5 x$

$= 2\log_5 5 + \log_5 x$

Substituting $y = \log_5 x$:

$= 2 + y$

d Let $z = \log_x 125$

$x^z = 125$

$\log_5 x^z = \log_5 125$

$z\log_5 x = \log_5 5^3$

$z\log_5 x = 3\log_5 5$

Substituting $y = \log_5 x$:

$zy = 3$

$z = \frac{3}{y}$

$\log_x 125 = \frac{3}{y}$

If you need to change bases, you can change the equation to exponential form and then use logarithms of a different base, as in worked solution **8 d**.

9 **b** $\log_4\left(\frac{2}{q}\right)$

$= \log_4 2 - \log_4 q$

Substituting $y = \log_4 q$:

$= \log_4 4^{\frac{1}{2}} - y$

$= \frac{1}{2}\log_4 4 - y$

$= \frac{1}{2} - y$

d $\log_4 p^2 - \log_4\left(4\sqrt{q}\right)$

$= 2\log_4 p - \left(\log_4 4 + \log_4 q^{\frac{1}{2}}\right)$

Substituting $x = \log_4 p$:

$= 2x - 1 - \frac{1}{2}\log_4 q$

Substituting $y = \log_4 q$:

$= 2x - 1 - \frac{1}{2}y$

10 **b** $\log_a\left(\frac{\sqrt{x}}{y^2}\right)$

$= \log_a \sqrt{x} - \log_a y^2$

$= \log_a x^{\frac{1}{2}} - 2\log_a y$

Substituting $\log_a y = 4$:

$= \frac{1}{2}\log_a x - 2 \times 4$

Substituting $\log_a x = 7$:

$= \frac{1}{2} \times 7 - 8$

$= \frac{7}{2} - \frac{16}{2}$

$= -\frac{9}{2}$

c $\log_a \left(x^2\sqrt{y}\right)$

$= \log_a x^2 + \log_a \sqrt{y}$

$= 2\log_a x + \log_a y^{\frac{1}{2}}$

Substituting $\log_a x = 7$:

$= 2 \times 7 + \frac{1}{2}\log_a y$

Substituting $\log_a y = 4$:

$= 14 + \frac{1}{2} \times 4$

$= 16$

11 $x + y = -\log_2 5$ .... [1]

$y + z = 2\log_2 5 - 3\log_3 2$ .... [2]

$x + z = \log_3 2 - \log_2 5$ .... [3]

[2] − [1]:

$y + z - (x + y) = 2\log_2 5 - 3\log_3 2 + \log_2 5$

$z - x = 3\log_2 5 - 3\log_3 2$ .... [4]

[3] + [4]:

$$x + z + z - x = \log_3 2 - \log_2 5 + 3\log_2 5 - 3\log_3 2$$
$$2z = -2\log_3 2 + 2\log_2 5$$

$z = \log_2 5 - \log_3 2$ .... [5]

Substituting [5] into [3]:

$x + (\log_2 5 - \log_3 2) = \log_3 2 - \log_2 5$

$x = \log_3 2 - \log_2 5 - \log_2 5 + \log_3 2$

$x = 2\log_3 2 - 2\log_2 5$

Substituting [5] into [2]:

$y + z = 2\log_2 5 - 3\log_3 2$

$y + \log_2 5 - \log_3 2 = 2\log_2 5 - 3\log_3 2$

$y = 2\log_2 5 - 3\log_3 2 + \log_3 2 - \log_2 5$

$y = \log_2 5 - 2\log_3 2$

EXERCISE 2D

1 **b** $\log_3 4x - \log_3 2 = \log_3 7$

$$\log_3\left(\frac{4x}{2}\right) = \log_3 7$$
$$\frac{4x}{2} = 7$$
$$2x = 7$$
$$x = \frac{7}{2} = 3.5$$

In this worked solution, as in the worked examples in the coursebook, a check is included. You should carry out a check yourself for any problem including the solutions of equations. Sometimes the values that you obtain can cause one side or both sides of the equation to be undefined. This is particularly true with logarithms, where you might produce solutions that would require a logarithm of a negative number.

Check:

When $x = \frac{7}{2}$:

$$\log_3 4\left(\frac{7}{2}\right) - \log_3 2$$
$$= \log_3 14 - \log_3 2$$
$$= \log_3\left(\frac{14}{2}\right)$$
$$= \log_3 7$$

This is a defined solution.

$x = \frac{7}{2}$ is a solution because both sides of the equation are defined and equal for this value.

d $\log_{10}(x-4) = 2\log_{10} 5 + \log_{10} 2$

$$\log_{10}(x-4) = \log_{10} 5^2 + \log_{10} 2$$
$$\log_{10}(x-4) = \log_{10}(25 \times 2)$$
$$x - 4 = 50$$
$$x = 54$$

A common error when handling logarithms is to assume that $\log(a+b) = \log a + \log b$. Be very careful to avoid this. Remember that the multiplication law for logarithms is $\log(ab) = \log a + \log b$.

2 **a** $\log_{10}(2x+9) - \log_{10} 5 = 1$

$$\log_{10}\left(\frac{2x+9}{5}\right) = 1$$

Writing each side as an exponent of 10:

$$\frac{2x+9}{5} = 10^1$$
$$2x + 9 = 50$$
$$2x = 41$$
$$x = \frac{41}{2} = 20.5$$

c $\log_2(5-2x) = 3 + \log_2(x+1)$

$$\log_2(5-2x) - \log_2(x+1) = 3$$
$$\log_2\left(\frac{5-2x}{x+1}\right) = 3$$

Writing each side as an exponent of 2:

$$\frac{5-2x}{x+1} = 2^3 = 8$$
$$5-2x = 8(x+1)$$
$$5-2x = 8x+8$$
$$10x = -3$$
$$x = -\frac{3}{10}$$

3 **a** $\log_2 x + \log_2(x-1) = \log_2 20$
$$\log_2[x(x-1)] = \log_2 20$$
$$x(x-1) = 20$$
$$x^2 - x - 20 = 0$$
$$(x-5)(x+4) = 0$$
$x = 5$ or $x = -4$

You can only take the logarithm of a positive number, so $\log_2(-4)$ is not defined and x cannot be -4.

Hence $x = 5$ is the only solution.

> Logarithmic equations often produce solutions that appear to be fine algebraically but then do not work if you substitute them back into the original equation. In particular, you must remember that you can only take logarithms of positive numbers. It is always best to check any solutions by substituting them back into the original equation.

d $3 + 2\log_2 x = \log_2(3-10x)$
$$\log_2(3-10x) - 2\log_2 x = 3$$
$$\log_2(3-10x) - \log_2 x^2 = 3$$
$$\log_2\left(\frac{3-10x}{x^2}\right) = 3$$

Writing each side as an exponent of 2:
$$\frac{3-10x}{x^2} = 2^3 = 8$$
$$3-10x = 8x^2$$
$$8x^2 + 10x - 3 = 0$$
$$(4x-1)(2x+3) = 0$$
$$x = \frac{1}{4} \text{ or } x = -\frac{3}{2}$$

You can only take the logarithm of a positive number, so $\log_2\left(-\frac{3}{2}\right)$ is not defined.

Hence $x = \frac{1}{4}$ is the only solution.

4 **b** $\log_x 36 + \log_x 4 = 2$
$$\log_x(36 \times 4) = 2$$
$$\log_x(144) = 2$$
$$x^2 = 144$$
$$x = \pm\sqrt{144} = \pm 12$$

Logarithms only exist for positive bases, so $x = 12$ is the only solution.

d $2\log_x 32 = 3 + 2\log_x 4$
$$\log_x 32^2 - \log_x 4^2 = 3$$
$$\log_x\left(\frac{1024}{16}\right) = 3$$
$$x^3 = 64$$
$$x = \sqrt[3]{64} = 4$$

5 **a** $(\log_2 x)^2 - 8\log_2 x + 15 = 0$
$$(\log_2 x - 5)(\log_2 x - 3) = 0$$
$\log_2 x = 5$ or $\log_2 x = 3$

$x = 2^5 = 32$ or $x = 2^3 = 8$

c $$(\log_5 x)^2 + \log_5(x^3) = 10$$
$$(\log_5 x)^2 + 3\log_5 x - 10 = 0$$
$$(\log_5 x + 5)(\log_5 x - 2) = 0$$
$$\log_5 x = -5 \quad \text{or} \quad \log_5 x = 2$$
$$x = 5^{-5} = \frac{1}{3125} \quad \text{or} \quad x = 5^2 = 25$$

6 **b** $$4^x = 2^y$$
$$(2^2)^x = 2^y$$
$$2^{2x} = 2^y$$
$$y = 2x \text{ .... [1]}$$
$$3\log_{10} y = \log_{10} x + \log_{10} 2$$
$$\log_{10} y^3 = \log_{10}(2x)$$
$$2x = y^3 \text{ .... [2]}$$

Substituting [1] into [2]:

$$2x = (2x)^3$$
$$2x = 8x^3$$
$$8x^3 - 2x = 0$$
$$2x(4x^2 - 1) = 0$$
$$2x(2x - 1)(2x + 1) = 0$$
$$x = 0, \frac{1}{2}, -\frac{1}{2}$$

$x > 0$ because you can only take the logarithm of a positive number.

$$x = \frac{1}{2}$$
$$y = 2x = 1$$

c $$\log_4(x - y) = 2\log_4 x$$
$$\log_4(x - y) = \log_4 x^2$$
$$x - y = x^2$$
$$y = x - x^2 \text{ .... [1]}$$
$$\log_4(x + y + 9) = 0$$
$$x + y + 9 = 4^0 = 1$$
$$y = -x - 8 \text{ .... [2]}$$

Equating [1] and [2]:

$$x - x^2 = -x - 8$$
$$x^2 - 2x - 8 = 0$$
$$(x - 4)(x + 2) = 0$$
$$x = 4 \text{ or } x = -2$$

$x > 0$ because you can only take the logarithm of a positive number.

$$x = 4$$

Then from [2]:

$$y = -4 - 8 = -12$$

7 $$\log_{10}(x^2 y) = 4$$
$$\log_{10} x^2 + \log_{10} y = 4$$
$$2\log_{10} x + \log_{10} y = 4 \text{ .... [1]}$$
$$\log_{10}\left(\frac{x^4}{y^3}\right) = 18$$
$$\log_{10} x^4 - \log_{10} y^3 = 18$$
$$4\log_{10} x - 3\log_{10} y = 18 \text{ .... [2]}$$

3[1] + [2]:

$$10\log_{10} x = 30$$
$$\log_{10} x = 3$$

Substituting into [1]:

$$2\,(3) + \log_{10} y = 4$$
$$\log_{10} y = 4 - 6 = -2$$

EXERCISE 2E

1

In the coursebook there is a Tip box that explains that 'log 6' means the same as '$\log_{10} 6$'. If you see a logarithm written as log without a base, then it generally means that it is a logarithm with base 10. However, you might see examples in some textbooks, or perhaps online, where '$\log x$' can mean the logarithm of any base relevant to the question. The important thing to remember is that you need to use the *same* base throughout the solution. Try some of these questions using a different base throughout. You will find that the answers are the same.

f $3^x = 2^{x+1}$

Taking logs to base 10 of both sides:

$$\begin{aligned}\log 3^x &= \log 2^{x+1}\\ x\log 3 &= (x+1)\log 2\\ x\log 3 &= x\log 2 + \log 2\\ x\log 3 - x\log 2 &= \log 2\\ x(\log 3 - \log 2) &= \log 2\\ x &= \frac{\log 2}{\log 3 - \log 2} = 1.71\end{aligned}$$

k $2^{x+1} = 3(5^x)$

Taking logs to base 10 of both sides and using the laws of logarithms:

$$\begin{aligned}(x+1)\log 2 &= \log 3 + \log 5^x\\ x\log 2 + \log 2 &= \log 3 + x\log 5\\ x(\log 2 - \log 5) &= \log 3 - \log 2\\ x &= \frac{\log 3 - \log 2}{\log 2 - \log 5} = -0.443\end{aligned}$$

2 **a**

$$\begin{aligned}2^{x+1} + 6(2^{x-1}) &= 12\\ 2^x \times 2^1 + 6\left(\frac{2^x}{2^1}\right) &= 12\\ 2(2^x) + \frac{6}{2}(2^x) &= 12\\ 2(2^x) + 3(2^x) &= 12\end{aligned}$$

b Using the result from part **a**:

$$\begin{aligned}2(2^x) + 3(2^x) &= 12\\ 5(2^x) &= 12\\ 2^x &= \frac{12}{5}\end{aligned}$$

Taking logs to base 10 of both sides:

$$\begin{aligned}\log 2^x &= \log\left(\frac{12}{5}\right)\\ x\log 2 &= \log 12 - \log 5\\ x &= \frac{\log 12 - \log 5}{\log 2} = 1.26\end{aligned}$$

3 **b**

$$\begin{aligned}3^{x+1} &= 3^{x-1} + 3^2\\ 3(3^x) &= \frac{3^x}{3} + 9\\ \left(3 - \frac{1}{3}\right)3^x &= 9\\ \frac{8}{3}(3^x) &= 9\\ 3^x &= \frac{3 \times 9}{8} = \frac{27}{8}\end{aligned}$$

Taking logs to base 10 of both sides:

$$\log(3^x) = \log\left(\frac{27}{8}\right)$$
$$x\log 3 = \log\left(\frac{27}{8}\right)$$
$$x = \frac{\log\left(\frac{27}{8}\right)}{\log 3} = 1.11$$

e $$4^{x-1} = 4^x - 4^3$$
$$\frac{4^x}{4} = 4^x - 64$$
$$\left(\frac{1}{4} - 1\right)4^x = -64$$
$$-\frac{3}{4}(4^x) = -64$$
$$4^x = \frac{64 \times 4}{3} = \frac{256}{3}$$

Taking logs to base 10 of both sides and using the laws of logarithms:

$$x\log 4 = \log\left(\frac{256}{3}\right)$$
$$x = \frac{\log\left(\frac{256}{3}\right)}{\log 4} = 3.21$$

4 $$2^{2x} + 32 = 12(2^x)$$
$$(2^x)^2 + 32 = 12(2^x)$$

Substituting $y = 2^x$:

$$y^2 + 32 = 12y$$
$$y^2 - 12y + 32 = 0$$
$$(y-8)(y-4) = 0$$

$$y = 8$$
$$2^x = 8$$
$$\log 2^x = \log 8$$
$$x\log 2 = \log 8$$
$$x = \frac{\log 8}{\log 2} = 3$$

or

$$y = 4$$
$$2^x = 4$$
$$\log 2^x = \log 4$$
$$x\log 2 = \log 4$$
$$x = \frac{\log 4}{\log 2} = 2$$

5 **b** $$2^{2x} + 5 = 6 \times 2^x$$
$$(2^x)^2 + 5 = 6(2^x)$$
$$(2^x)^2 - 6(2^x) + 5 = 0$$

Let $y = 2^x$:

$$y^2 - 6y + 5 = 0$$
$$(y-5)(y-1) = 0$$

$$y = 5$$
$$2^x = 5$$
$$\log(2^x) = \log 5$$
$$x\log 2 = \log 5$$
$$x = \frac{\log 5}{\log 2} = 2.32$$

or

$$y = 1$$
$$2^x = 1$$
$$\log(2^x) = \log 1$$
$$x\log 2 = 0$$
$$x = 0$$

d $$4^{2x} + 27 = 12(4^x)$$
$$(4^x)^2 + 27 = 12(4^x)$$
$$(4^x)^2 - 12(4^x) + 27 = 0$$

Let $y = 4^x$:

$$y^2 - 12y + 27 = 0$$
$$(y-9)(y-3) = 0$$

or

$$\begin{aligned} y &= 9 \\ 4^x &= 9 \\ \log 4^x &= \log 9 \\ x\log 4 &= \log 9 \\ x &= \frac{\log 9}{\log 4} = 1.58 \end{aligned} \qquad \begin{aligned} y &= 3 \\ 4^x &= 3 \\ \log 4^x &= \log 3 \\ x\log 4 &= \log 3 \\ x &= \frac{\log 3}{\log 4} = 0.792 \end{aligned}$$

6

$$\begin{aligned} 2^{2x} - 5(2^{x+1}) + 24 &= 0 \\ (2^x)^2 - 5(2 \times 2^x) + 24 &= 0 \\ (2^x)^2 - 10(2^x) + 24 &= 0 \end{aligned}$$

Substituting $u = 2^x$:

$$\begin{aligned} u^2 - 10u + 24 &= 0 \\ (u-6)(u-4) &= 0 \end{aligned}$$

$$\begin{aligned} u &= 6 \\ 2^x &= 6 \\ \log 2^x &= \log 6 \\ x\log 2 &= \log 6 \\ x &= \frac{\log 6}{\log 2} = 2.58 \end{aligned} \qquad \text{or} \qquad \begin{aligned} u &= 4 \\ 2^x &= 4 \\ \log(2^x) &= \log 4 \\ x\log 2 &= \log 4 \\ x &= \frac{\log 4}{\log 2} = 2 \end{aligned}$$

7 **b**

$$\begin{aligned} 3^{2x} - 3^{x+1} &= 10 \\ (3^x)^2 - 3(3^x) &= 10 \end{aligned}$$

Let $u = 3^x$:

$$\begin{aligned} u^2 - 3u - 10 &= 0 \\ (u-5)(u+2) &= 0 \\ u &= 5 \\ 3^x &= 5 \\ \log(3^x) &= \log 5 \\ x\log 3 &= \log 5 \\ x &= \frac{\log 5}{\log 3} \\ &= 1.46 \end{aligned}$$

Or

$$\begin{aligned} u &= -2 \\ 3^x &= -2 \end{aligned}$$

No real solutions, because $3^x > 0$ for any x.

d

$$\begin{aligned} 4^{2x+1} &= 17(4^x) - 15 \\ 4(4^{2x}) - 17(4^x) + 15 &= 0 \\ 4(4^x)^2 - 17(4^x) + 15 &= 0 \end{aligned}$$

Let $u = 4^x$:

$$\begin{aligned} 4u^2 - 17u + 15 &= 0 \\ (4u-5)(u-3) &= 0 \end{aligned}$$

$$\begin{aligned} u &= \frac{5}{4} \\ 4^x &= \frac{5}{4} \\ \log(4^x) &= \log\left(\frac{5}{4}\right) \\ x\log 4 &= \log\left(\frac{5}{4}\right) \\ x &= \frac{\log\left(\frac{5}{4}\right)}{\log 4} \\ &= 0.161 \end{aligned} \qquad \text{or} \qquad \begin{aligned} u &= 3 \\ 4^x &= 3 \\ \log(4^x) &= \log 3 \\ x\log 4 &= \log 3 \\ x &= \frac{\log 3}{\log 4} = 0.792 \end{aligned}$$

8 **a** $4^x + 2^x - 12 = 0$

$(2^2)^x + 2^x - 12 = 0$

$2^{2x} + 2^x - 12 = 0$

$(2^x)^2 + 2^x - 12 = 0$

Let $u = 2^x$:

$u^2 + u - 12 = 0$

$(u+4)(u-3) = 0$

$u = -4$

$2^x = -4$

No real solutions because $2^x > 0$.

Or $u = 3$

$2^x = 3$

$\log(2^x) = \log 3$

$x \log 2 = \log 3$

$x = \dfrac{\log 3}{\log 2} = 1.58$

When an equation includes *different* bases raised to the power of x, it is useful to consider whether or not those bases are powers of the same number. In Question **8 a** both 4 and 2 are powers of 2 and this leads to the solution.

d $2(9^x) = 3^{x+1} + 27$

$2(3^2)^x = 3^{x+1} + 27$

$2(3^{2x}) - 3^{x+1} - 27 = 0$

$2(3^x)^2 - 3(3^x) - 27 = 0$

Let $u = 3^x$.

$2u^2 - 3u - 27 = 0$

$(2u - 9)(u + 3) = 0$

$u = \dfrac{9}{2}$

$3^x = \dfrac{9}{2}$

$\log(3^x) = \log\left(\dfrac{9}{2}\right)$

$x \log 3 = \log\left(\dfrac{9}{2}\right)$

$x = \dfrac{\log\left(\dfrac{9}{2}\right)}{\log 3}$

$= 1.37$

Or

$u = -3$

$3^x = -3$

No real solutions because $3^x > 0$.

9 **a** $3^x = 7^y$

Taking logs to base 10 to both sides:

$\log(3^x) = \log(7^y)$

$x \log 3 = y \log 7$

$\dfrac{x}{y} = \dfrac{\log 7}{\log 3} = 1.77$

b $7^x = (2.7)^y$

Taking logs to base 10 of both sides:

$$\log 7^x = \log (2.7)^y$$
$$x \log 7 = y \log 2.7$$
$$\frac{x}{y} = \frac{\log 2.7}{\log 7} = 0.510$$

c $4^{2x} = 3^{5y}$

Taking logs to base 10 of both sides:

$$\log 4^{2x} = \log 3^{5y}$$
$$2x \log 4 = 5y \log 3$$
$$x(2\log 4) = y(5\log 3)$$
$$\frac{x}{y} = \frac{5\log 3}{2\log 4} = 1.98$$

10 $x^{2.5} = 20x^{1.25}$

$$x^{2.5} - 20x^{1.25} = 0$$
$$x^{1.25}(x^{1.25} - 20) = 0$$

$x^{1.25} = 0$ or $x^{\frac{5}{4}} = 20$

$x = 0$

$$\log_{20} x^{\frac{5}{4}} = \log_{20} 20$$
$$\frac{5}{4}\log_{20} x = 1$$
$$\log_{20} x = \frac{4}{5}$$
$$x = 20^{\frac{4}{5}} = 20^{0.8}$$

Remember that, when solving equations, you must not divide through by anything that can be zero. If you do you will often lose a solution. Instead, it is important that you factorise, as in worked solution **10**.

11 **c** $3\,|2^x - 5| = 2^x$

$$3(2^x - 5) = 2^x$$
$$3(2^x) - 15 = 2^x$$
$$2(2^x) = 15$$
$$2^x = \frac{15}{2}$$
$$\log 2^x = \log\left(\frac{15}{2}\right)$$
$$x\log 2 = \log\left(\frac{15}{2}\right)$$
$$x = \frac{\log\left(\frac{15}{2}\right)}{\log 2} = 2.91$$

or

$$3(5 - 2^x) = 2^x$$
$$15 - 3(2^x) = 2^x$$
$$4(2^x) = 15$$
$$2^x = \frac{15}{4}$$
$$\log(2^x) = \log\left(\frac{15}{4}\right)$$
$$x\log 2 = \log\left(\frac{15}{4}\right)$$
$$x = \frac{\log(\frac{15}{4})}{\log 2}$$
$$x = 1.91$$

g $3^{2|x|} = 6\left(3^{|x|}\right) + 16$

$$\left(3^{|x|}\right)^2 - 6(3^{|x|}) - 16 = 0$$

Let $u = 3^{|x|}$:

$$u^2 - 6u - 16 = 0$$
$$(u - 8)(u + 2) = 0$$
$$u = 8$$
$$3^{|x|} = 8$$
$$\log 3^{|x|} = \log 8$$
$$|x|\log 3 = \log 8$$
$$|x| = \frac{\log 8}{\log 3}$$
$$x = \pm\frac{\log 8}{\log 3} = \pm 1.89$$

Or

$$u = -2$$
$$3^{|x|} = -2$$

No real solutions as $3^{|x|} > 0$.

12 **a** $2^{4x+1} \times 3^{2-x} = 8^x \times 3^{5-2x}$

$$\frac{2^{4x+1}}{8^x} = \frac{3^{5-2x}}{3^{2-x}}$$
$$\frac{2^{4x+1}}{(2^3)^x} = 3^{5-2x-2+x}$$
$$2^{4x+1-3x} = 3^{3-x}$$
$$2^{x+1} = 3^{3-x}$$
$$\frac{2^{x+1}}{3^{3-x}} = 1$$
$$2(2^x)(3^{x-3}) = 1$$
$$2(2^x)\frac{(3^x)}{3^3} = 1$$
$$(2^x)(3^x) = \frac{27}{2}$$
$$6^x = \frac{27}{2} = 13.5$$

b Using the result from part **a**:

$6^x = 13.5$

Taking logs to base 10 of both sides:

$$\log(6^x) = \log(13.5)$$
$$x \log 6 = \log 13.5$$
$$x = \frac{\log 13.5}{\log 6} = 1.45$$

13

> Not all questions require the use of logs. If you can find a solution using a method that you are more comfortable with then there is no problem in doing so. In Question **13** logarithms are difficult to use until the very end because the equation involves sums and differences of terms; log rules do not include $\log(A + B)$ or $\log(A - B)$.

$$8\left(8^{x-1} - 1\right) = 7\left(4^x - 2^{x+1}\right)$$
$$8\left(2^3\right)^{x-1} - 8 = 7\left(2^2\right)^x - 7\left(2 \times 2^x\right)$$
$$8\left(2^{3x-3}\right) - 8 = 7\left(2^{2x}\right) - 14\left(2^x\right)$$
$$\frac{8\left(2^{3x}\right)}{2^3} - 8 = 7(2^x)^2 - 14\left(2^x\right)$$
$$(2^x)^3 - 8 = 7(2^x)^2 - 14\left(2^x\right)$$
$$(2^x)^3 - 7(2^x)^2 + 14\left(2^x\right) - 8 = 0$$

Letting $u = 2^x$:

$$u^3 - 7u^2 + 14u - 8 = 0$$
$$(2)^3 - 7(2)^2 + 14(2) - 8 = 0$$

So $u - 2$ is a factor of $u^3 - 7u^2 + 14u - 8$ by the factor theorem.

Dividing:

$$\begin{array}{r} u^2 - 5u + 4 \\ u - 2\overline{)u^3 - 7u^2 + 14u - 8} \\ \underline{u^3 - 2u^2} \qquad\qquad \\ -5u^2 + 14u \quad\;\; \\ \underline{-5u^2 + 10u} \quad\;\; \\ 4u - 8 \\ \underline{4u - 8} \\ 0 \end{array}$$

$$u^3 - 7u^2 + 14u - 8 = (u-2)\left(u^2 - 5u + 4\right) = 0$$

$$(u-2)(u-4)(u-1) = 0$$

$$u = 2 \quad \text{or} \quad u = 4 \quad \text{or} \quad u = 1$$

$$2^x = 2 \text{ or } 2^x = 4 \text{ or } 2^x = 1$$

$$x = 1 \text{ or } x = 2 \text{ or } x = 0$$

EXERCISE 2F

1 **a** $2^x < 5$

Taking logs to base 10 of both sides:

$$\begin{aligned} \log 2^x &< \log 5 \\ x \log 2 &< \log 5 \\ x &< \frac{\log 5}{\log 2} \end{aligned}$$

c $\left(\frac{2}{3}\right)^x > 3$

Taking logs to base 10 of both sides:

$$\log \left(\frac{2}{3}\right)^x > \log 3$$

$$x \log \left(\frac{2}{3}\right) > \log 3$$

Dividing through by $\log \left(\frac{2}{3}\right)$ and reversing the inequality symbol because $\log \left(\frac{2}{3}\right) < 0$:

$$x < \frac{\log 3}{\log \left(\frac{2}{3}\right)}$$

> Remember that the logarithm of any number less than 1 will be negative. If you divide both sides by a negative number you need to reverse the direction of the inequality.

2 **b** $3^{2x+5} > 20$

Taking logs to base 10 of both sides and then using the laws of logarithms:

$$\begin{aligned} \log 3^{2x+5} &> \log\ 20 \\ (2x+5) \log\ 3 &> \log\ 20 \\ 2x+5 &> \frac{\log\ 20}{\log\ 3} \\ 2x &> \frac{\log\ 20}{\log\ 3} - 5 \\ x &> \frac{1}{2}\left(\frac{\log\ 20}{\log\ 3} - 5\right) \end{aligned}$$

d $7 \times \left(\frac{5}{6}\right)^{3-x} > 4$

$$\left(\frac{5}{6}\right)^{3-x} > \frac{4}{7}$$

Taking logs to base 10 of both sides and then using the laws of logarithms:

$$\log \left(\frac{5}{6}\right)^{3-x} > \log \frac{4}{7}$$

$$(3-x) \log \left(\frac{5}{6}\right) > \log \frac{4}{7}$$

Dividing through by $\log \left(\frac{5}{6}\right)$ and reversing the inequality symbol because $\log \left(\frac{5}{6}\right) < 0$:

$$3 - x < \frac{\log \frac{4}{7}}{\log \frac{5}{6}}$$

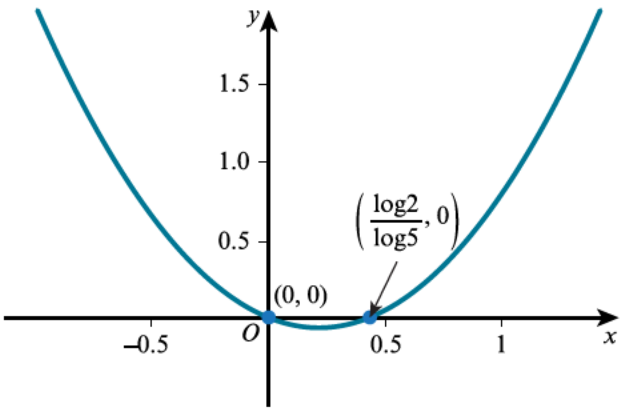
$$x > 3 - \frac{\log \frac{4}{7}}{\log \frac{5}{6}}$$

3 $5^{x^2} > 2^x$

Taking logs to base 10 of both sides:

$$\log 5^{x^2} > \log 2^x$$
$$x^2 \log 5 > x \log 2$$
$$x^2 \log 5 - x \log 2 > 0$$
$$x(x \log 5 - \log 2) > 0$$

Critical values are $x = 0$ and $x = \dfrac{\log 2}{\log 5}$.



$x < 0$ or $x > \dfrac{\log 2}{\log 5}$

4 $3^{2x-1} \times 2^{1-3x} \geqslant 5$

Taking logs to base 10 of both sides and using the laws of logarithms:

$$\log 3^{2x-1} + \log 2^{1-3x} \geqslant \log 5$$
$$(2x-1)\log 3 + (1-3x)\log 2 \geqslant \log 5$$
$$2x \log 3 - \log 3 + \log 2 - 3x \log 2 \geqslant \log 5$$
$$x(2\log 3 - 3\log 2) \geqslant \log 5 + \log 3 - \log 2$$
$$x(\log \frac{3^2}{2^3}) \geqslant \log \frac{15}{2}$$
$$x \log\left(\frac{9}{8}\right) \geqslant \log\left(\frac{15}{2}\right)$$
$$x \geqslant \frac{\log\left(\frac{15}{2}\right)}{\log\left(\frac{9}{8}\right)}$$

5 **a** $\log_{10} 4 = 0.60206$

$$\log_{10} 4^{100} = 100 \log_{10} 4 = 60.206$$
$$4^{100} = 10^{60.206} = 10^{60} \times 10^{0.206}$$

Using the fact that $10^{0.206} < 2$:

$4^{100} < 2 \times 10^{60}$

The number 2×10^{60} consists of the digit 2 followed by 60 zeros.

It has 61 digits.

b Given that in standard form $4^{100} < 2 \times 10^{60}$ from part **a**, this must mean that $4^{100} = 1.a \times 10^{60}$, where a is a decimal number less than 1.

Therefore, the first digit is 1.

EXERCISE 2G

Calculators do not all use the same symbols for 'to the power of'. Amongst others, the symbols commonly used are: ^, x^y and $x^{\square}$. For e^x you may even find $e^{\square}$.

In the worked solutions shown here, x^y is used as an example. Make sure you know how to use the equivalent key on your calculator.

1 **b** $\boxed{e}\ \boxed{x^y}\ \boxed{2}\ \boxed{.}\ \boxed{7}\ \boxed{=}\ 14.9$

d $\boxed{e}\ \boxed{x^y}\ \boxed{-}\ \boxed{2}\ \boxed{=}\ 0.135$

2 **b** $\boxed{\ln}\ \boxed{1}\ \boxed{.}\ \boxed{4}\ \boxed{=}\ 0.336$

d $\boxed{\ln}\ \boxed{0}\ \boxed{.}\ \boxed{1}\ \boxed{5}\ \boxed{=}\ -1.90$

3 **b** $e^{\frac{1}{2}\ln 9}$

$= e^{\ln\left(9^{\frac{1}{2}}\right)}$

$= e^{\ln\sqrt{9}}$

$= e^{\ln 3}$

$= 3$

d $e^{-\ln\frac{1}{2}}$

$= e^{\ln\left(\frac{1}{2}\right)^{-1}}$

$= \left(\frac{1}{2}\right)^{-1}$

$= \frac{2}{1} = 2$

4 **b** $\ln e^x = 15$

$x = 15$

d $e^{-\ln x} = 3$

$e^{\ln x^{-1}} = 3$

$x^{-1} = 3$

$\frac{1}{x} = 3$

$x = \frac{1}{3}$

5 **b** $e^{2x} = 25$

Taking natural logs of both sides:

$\ln e^{2x} = \ln 25$

$2x = \ln 25$

$x = \frac{\ln 25}{2} = 1.61$

d $e^{2x-3} = 16$

Taking natural logs of both sides:

$\ln e^{2x-3} = \ln 16$

$2x - 3 = \ln 16$

$2x = \ln 16 + 3$

$x = \frac{1}{2}(\ln 16 + 3) = 2.89$

6 **b** $e^{3x} = 7$

Taking natural logs of both sides:

$$\ln e^{3x} = \ln 7$$
$$3x = \ln 7$$
$$x = \frac{1}{3}\ln 7$$

Recall that $\ln x$ and e^x are inverses of one another. Although the logarithm has been written on both sides in each of these worked solutions, you can simply write 'x'. You don't need to include the step involving the expression '$\ln e^x$'.

d $e^{\frac{1}{2}x+3} = 4$

Taking natural logs of both sides:

$$\ln e^{\frac{1}{2}x+3} = \ln 4$$
$$\frac{1}{2}x + 3 = \ln 4$$
$$\frac{1}{2}x = \ln 4 - 3$$
$$x = 2(\ln 4 - 3)$$

7 **a** $e^x > 10$

Taking natural logs of both sides:

$$\ln e^x > \ln 10$$
$$x > \ln 10$$

b $e^{5x-2} \leqslant 35$

Taking natural logs of both sides:

$$\ln e^{5x-2} \leqslant \ln 35$$
$$5x - 2 \leqslant \ln 35$$
$$5x \leqslant \ln 35 + 2$$
$$x \leqslant \frac{1}{5}(\ln 35 + 2)$$

c $5 \times e^{2x+3} < 1$

$$e^{2x+3} < \frac{1}{5}$$

Taking natural logs of both sides:

$$\ln e^{2x+3} < \ln\left(\frac{1}{5}\right)$$
$$2x + 3 < \ln\left(\frac{1}{5}\right)$$
$$2x < \ln\left(\frac{1}{5}\right) - 3$$
$$x < \frac{1}{2}\left(\ln\left(\frac{1}{5}\right) - 3\right)$$

8 **a** $\ln x = 5$

Writing each side as an exponential of e:

$$e^{\ln x} = e^5$$
$$x = e^5 = 148$$

c $\ln(x - 2) = 6$

Writing each side as an exponential of e:

$$e^{\ln(x-2)} = e^6$$
$$x - 2 = e^6$$
$$x = 2 + e^6 = 405$$

9 **b** $2\ln(x+2) - \ln x = \ln(2x-1)$

$$\ln(x+2)^2 - \ln x = \ln(2x-1)$$
$$\ln\frac{(x+2)^2}{x} = \ln(2x-1)$$
$$\frac{(x+2)^2}{x} = 2x-1$$
$$(x+2)^2 = 2x^2 - x$$
$$x^2 + 4x + 4 = 2x^2 - x$$
$$x^2 - 5x - 4 = 0$$
$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4\times 1\times(-4)}}{2\times 1}$$
$$= 5.70 \text{ or } -0.702$$

But neither $x+2$ or $2x-1$ can be negative,

so $x = 5.70$ is the only solution.

> Before you start each question, look at the equations involved. Are they valid for all values of x? Are you likely to need to choose between solutions, particularly those that are positive or negative?

f $\ln(2+x^2) = 1 + 2\ln x$

Writing each side as an exponential of e:

$$e^{\ln(2+x^2)} = e^{1+2\ln x}$$
$$2 + x^2 = e \times e^{2\ln x}$$
$$2 + x^2 = e \times e^{\ln x^2}$$
$$2 + x^2 = ex^2$$
$$2 + x^2 = ex^2$$
$$(e-1)x^2 = 2$$
$$x^2 = \frac{2}{e-1}$$

$x > 0$ because the original equation involves a logarithm of x.

There, the only solution is $x = \sqrt{\frac{2}{e-1}} = 1.08$.

10 **a** $2\ln(y+1) - \ln y = \ln(x+y)$

$$\ln(y+1)^2 - \ln y = \ln(x+y)$$
$$\ln\frac{(y+1)^2}{y} = \ln(x+y)$$
$$\frac{(y+1)^2}{y} = x+y$$
$$(y+1)^2 = xy + y^2$$
$$y^2 + 2y + 1 = xy + y^2$$
$$xy - 2y = 1$$
$$y(x-2) = 1$$
$$y = \frac{1}{x-2}$$

b $\ln(y+2) - \ln y = 1 + 2\ln x$

$$\ln\frac{y+2}{y} = 1 + \ln x^2$$

Writing each side as an exponential of e:

$$\frac{y+2}{y} = e^{1+\ln x^2}$$

$$\frac{y+2}{y} = e \times e^{\ln x^2}$$

$$\frac{y+2}{y} = ex^2$$

$$y + 2 = ex^2 y$$

$$ex^2 y - y = 2$$

$$y(ex^2 - 1) = 2$$

$$y = \frac{2}{ex^2 - 1}$$

11 **a** $e^{2x} + 2e^x - 15 = 0$

$(e^x)^2 + 2e^x - 15 = 0$

Let $u = e^x$.

$u^2 + 2u - 15 = 0$

$(u+5)(u-3) = 0$

$u = -5$

$e^x = -5$

No real solutions because $e^x > 0$.

or $u = 3$

$e^x = 3$

So $x = \ln 3$ is the only solution.

> Remember that any power of a positive number will always be positive. A common error is to assume that a negative power will produce a negative value.

c $6e^{2x} - 13e^x - 5 = 0$

$6(e^x)^2 - 13e^x - 5 = 0$

Let $u = e^x$.

$6u^2 - 13u - 5 = 0$

$(3u+1)(2u-5) = 0$

$u = -\frac{1}{3}$

$e^x = -\frac{1}{3}$

No real solutions because $e^x > 0$.

or $u = \frac{5}{2}$

$e^x = \frac{5}{2}$

so $x = \ln\left(\frac{5}{2}\right)$ is the only solution.

12 For simplicity, writing y instead of $f(x)$:

$y = 5e^x + 2$

$5e^x = y - 2$

$e^x = \frac{y-2}{5}$

Taking natural logs of both sides:

$x = \ln\left(\frac{y-2}{5}\right)$

Swapping x and y:

$f^{-1}(x) = \ln\left(\frac{x-2}{5}\right)$

13 $2\ln(3-e^{2x})=1$

$\ln(3-e^{2x})=\frac{1}{2}$

Writing both sides as an exponential of e:

$3-e^{2x}=e^{\frac{1}{2}}$

$e^{2x}=3-e^{\frac{1}{2}}$

Taking natural logs of both sides:

$2x=\ln\left(3-e^{\frac{1}{2}}\right)$

$x=\frac{1}{2}\ln(3-e^{\frac{1}{2}})=0.151$

14 **a** $2\ln x+\ln y=1+\ln 5$ .... [1]

$\ln 10x-\ln y=2+\ln 2$ .... [2]

[1] + [2]:

$2\ln x+\ln 10x=3+\ln 2+\ln 5$

$\ln x^2+\ln 10x=3+\ln 10$

$\ln(10x^3)-\ln 10=3$

$\ln\frac{10x^3}{10}=3$

$\ln x^3=3$

$3\ln x=3$

$\ln x=1$

$x=e^1=e$

Substituting into [1]:

$2\ln e+\ln y=1+\ln 5$

$\ln y=1+\ln 5-2\ln e$

$\ln y=1+\ln 5-2$

$\ln y=\ln 5-1$

$\ln 5-\ln y=1$

$\ln\frac{5}{y}=1$

$\frac{5}{y}=e^1=e$

$y=\frac{5}{e}$

As an exercise, check that these solutions work in both of the original equations in parts **a** and **b** of Question **14**.

b $e^{3x+4y}=2e^{2x-y}$ .... [1]

$e^{2x+y}=8e^{x+6y}$ .... [2]

Dividing through by e^{2x-y} in [1].

$e^{3x+4y-2x+y}=2$

$e^{x+5y}=2$

Taking natural logs of both sides:

$x+5y=\ln 2$ .... [3]

Dividing through by e^{x+6y} in [2].

$e^{2x+y-x-6y}=8$

$e^{x-5y}=8$

Taking natural logs of both sides:

$x-5y=\ln 8$ .... [4]

[3] + [4]:

$$2x = \ln 2 + \ln 8$$
$$x = \frac{1}{2}\ln 16 = \ln 16^{\frac{1}{2}} = \ln \sqrt{16} = \ln 4$$

Substituting into [3]:

$$\ln 4 + 5y = \ln 2$$
$$5y = \ln 2 - \ln 4$$
$$y = \frac{1}{5}\ln\left(\frac{1}{2}\right)$$

15 $\ln(2x + 1) \leqslant \ln(x + 4)$

$2x + 1 \leqslant x + 4$

$x \leqslant 3$

But this is not the end of the solution.

Remember that you can only take the logarithm of a positive number.

So $2x + 1$ and $x + 4$ must also be positive.

$$2x + 1 > 0$$
$$2x > -1$$
$$x > -\frac{1}{2}$$

$$x + 4 > 0$$
$$x > -4$$

Giving $x > -\frac{1}{2}$.

So, the solution is:

$$-\frac{1}{2} < x \leqslant 3$$

EXERCISE 2H

1 **b** $y = 10^{ax-b}$

Taking logs to base 10 of both sides:

$$\log y = \log 10^{ax-b}$$
$$\log y = (ax - b)\log 10$$
$$\log y = ax - b$$
$$Y = mX + c$$
$$Y = \log y$$
$$m = a$$
$$X = x$$
$$c = -b$$

It is always important to compare your equation with $Y = mX + c$ and write down what Y, m, X and c represent.

E

e $a = e^{x^2+by}$

Taking natural logs of both sides:

$$\ln a = x^2 + by$$
$$by = -x^2 + \ln a$$
$$y = -\frac{1}{b}x^2 + \frac{1}{b}\ln a$$
$$Y = y$$
$$m = -\frac{1}{b}$$
$$X = x^2$$
$$c = \frac{1}{b}\ln a$$

Note that this answer appears to be different to the answer given in the coursebook. This is because in this worked solution we have chosen to keep Y and y on the same axis. Also X and x are on the same axis. The coursebook version uses $Y = x^2$, effectively swapping axes.

2 $y = ax^n$

Taking natural logs of both sides:

$$\ln y = \ln a + \ln x^n$$
$$\ln y = n\ln x + \ln a$$
$$Y = \ln y$$
$$m = n$$
$$X = \ln x$$
$$c = \ln a$$

From the diagram:

$$m = \frac{4.02 - 3.22}{0.31 - 1.83} = -0.53 = n$$
$$Y = -0.526.\ldots X + c$$
$$4.02 = -0.526.\ldots \times 0.31 + c$$
$$c = 4.02 + 0.526.\ldots \times 0.31 = 4.183\ \ldots.$$
$$\ln a = 4.183\ \ldots.$$
$$a = e^{4.183\ldots.} = 66$$

3 $y = k \times e^{n(x-2)}$

Taking natural logs of both sides:

$$\ln y = \ln k + \ln\ e^{n(x-2)}$$
$$\ln y = \ln k + n(x-2)$$
$$\ln y = nx + \ln k - 2n$$
$$Y = \ln y$$
$$m = n$$
$$X = x$$
$$c = \ln k - 2n$$

From the diagram:

$$\text{Gradient} = \frac{4.33 - 1.84}{7-1} = 0.415$$
$$n = 0.42 (\text{to 2 significant figures})$$
$$Y = mX + c$$
$$1.84 = 0.415X + c$$
$$1.84 = 0.415 \times 1 + c$$
$$c = 1.84 - 0.415 = 1.425$$
$$c = \ln k - 2n$$
$$1.425 = \ln k - 2 \times 0.415$$
$$\ln k = 1.425 + 2 \times 0.415 = 2.255$$
$$k = e^{2.255} = 9.5$$

4 **a** $\log_{10} y = mx + c$

$$m = \frac{11-5}{6-2} = \frac{6}{4} = \frac{3}{2}$$
$$\log_{10} y = \frac{3}{2}x + c$$
$$5 = \frac{3}{2} \times 2 + c$$
$$5 = 3 + c$$
$$c = 2$$
$$\log_{10} y = \frac{3}{2}x + 2$$

b Using the result from part **a**:

$$\log_{10} y = \frac{3}{2}x + 2$$

Writing both sides as an exponential of 10:

$$y = 10^{\frac{3}{2}x+2}$$
$$y = 10^2 \times 10^{\frac{3}{2}x}$$
$$y = 100 \times 10^{\frac{3}{2}x}$$

5 **a** $\ln y = m \ln x + c$

$$m = \frac{13-4}{5-2} = 3$$
$$\ln y = 3 \ln x + c$$
$$4 = 3 \times 2 + c$$
$$c = -2$$
$$\ln y = 3 \ln x - 2$$

b Using the result from part **a**:

$$\ln y = 3 \ln x - 2$$

Writing both sides as an exponential of e:

$$y = e^{\ln x^3 - 2}$$
$$y = e^{-2} \times e^{\ln x^3}$$
$$y = e^{-2}x^3 = \frac{x^3}{e^2}$$

6 $5^{2y} = 3^{2x+1}$

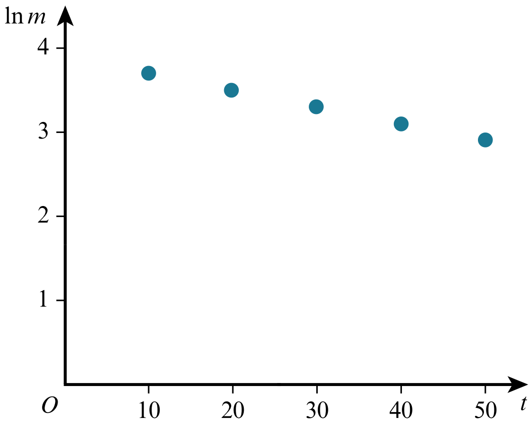
Taking natural logs of both sides:

$$\ln 5^{2y} = \ln 3^{2x+1}$$
$$2y \ln 5 = (2x+1)\ln 3$$
$$2y \ln 5 = (2\ln 3)x + \ln 3$$
$$y = \left(\frac{\ln 3}{\ln 5}\right)x + \frac{\ln 3}{2\ln 5}$$

$$\text{Gradient} = \frac{\ln 3}{\ln 5}$$

Crosses y-axis at $\left(0, \dfrac{\ln 3}{2\ln 5}\right)$.

7 **a**



b $m = m_0 e^{-kt}$

Taking natural logs of both sides:

$$\ln m = \ln m_0 + \ln e^{-kt}$$
$$\ln m = \ln m_0 - kt$$
$$\ln m = -kt + \ln m_0$$
$$Y = \ln m$$
$$m = -k$$
$$X = t$$
$$c = \ln m_0$$

From the graph is part **a**:

$$-k = \text{gradient} \approx \frac{2.91 - 3.71}{50 - 10} = -0.02$$
$$k = 0.02$$
$$c \approx 3.91$$
$$\ln m_0 = 3.91$$
$$m_0 = e^{3.91} = 49.9$$

c $m = m_0 e^{-kt}$

Substituting in the values found in part **b**:

$$m = 49.9e^{-0.02t}$$
$$\text{Original mass} = 49.9e^0 = 49.9$$
$$\text{Half mass} = \frac{49.9}{2} = 24.95$$
$$24.95 = 49.9e^{-0.02t}$$
$$e^{-0.02t} = \frac{1}{2}$$

Taking natural logs of both sides:

$$-0.02t = \ln \frac{1}{2}$$
$$t = -\frac{1}{0.02}\ln \frac{1}{2} = 34.7\text{days} \approx 35 \text{ days}$$

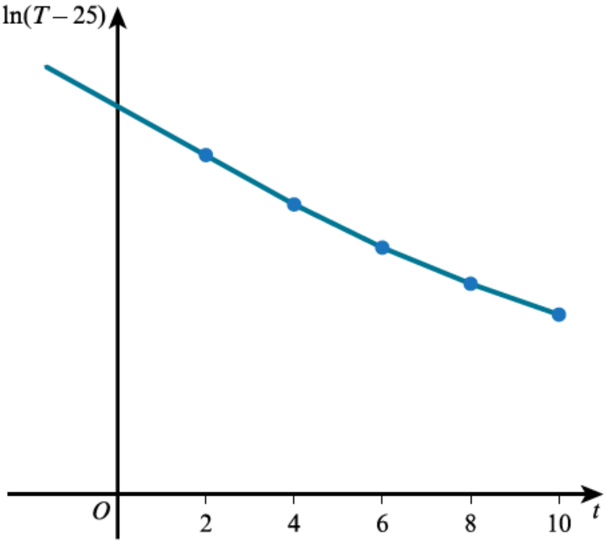
8 **a** $T = 25 + ke^{-nt}$

$$T - 25 = ke^{-nt}$$

Taking natural logs of both sides:

$$\begin{aligned}\ln(T-25) &= \ln k + \ln\ e^{-nt}\\ \ln(T-25) &= -nt + \ln k\\ Y &= \ln(T-25)\\ m &= -n\\ X &= t\\ c &= \ln k\end{aligned}$$

In part **a** you need to subtract 25 from both sides first because you cannot take a logarithm across the sum of two terms. This is not possible because $\log(a+b)$ is **not** the same as $\log a + \log b$.

b 

From the graph:

$$\text{Gradient} \approx \frac{3.01-3.65}{10-2} = -0.08$$

$$-n = -0.08$$

$$n = 0.08$$

$$y - \text{intercept} \approx 3.81$$

$$\ln k = 3.81$$

$$k \approx e^{3.81} = 45.1 \approx 45$$

c Substituting in the values from part **b**:

$T = 25 + 45e^{-0.08t}$

i $T(0) = 25 + 45e^{0} = 70\,^{\circ}\text{C}$

ii

$$28 = 25 + 45e^{-0.08t}$$

$$45e^{-0.08t} = 3$$

$$e^{-0.08t} = \frac{3}{45}$$

Taking natural logs of both sides:

$$-0.08t = \ln\left(\frac{3}{45}\right)$$

$$t = -\frac{1}{0.08}\ln\left(\frac{3}{45}\right) \approx 34\text{ min}$$

iii As t gets larger and larger, $e^{-0.08t} = \dfrac{1}{e^{0.08t}}$ gets smaller and smaller, tending to zero.

Room temperature is, therefore, $25 + 0 = 25\,^{\circ}\text{C}$.

To write, '$e^{-0.08t}$ tends to zero as t tends to infinity' mathematically, you write:

$e^{-0.08t} \to 0$ as $t \to \infty$.

END-OF-CHAPTER REVIEW EXERCISE 2

1 $2^x > 7$

Taking logs to base 10 of both sides:

$$\begin{aligned} \log 2^x &> \log 7 \\ x \log 2 &> \log 7 \\ x &> \frac{\log 7}{\log 2} \end{aligned}$$

2
$$\begin{aligned} \ln p &= 2 \ln q - \ln(3+q) \\ \ln p &= \ln q^2 - \ln(3+q) \\ \ln p &= \ln\left(\frac{q^2}{3+q}\right) \\ p &= \frac{q^2}{3+q} \end{aligned}$$

3
$$\begin{aligned} 3 \times 2^{3x+2} &< 8 \\ 3 \times 2^2 \times (2^3)^x &< 8 \\ 3 \times 8^x &< 2 \\ 8^x &< \frac{2}{3} \end{aligned}$$

Taking logs to base 10 of both sides:

$$\begin{aligned} \log 8^x &< \log\left(\frac{2}{3}\right) \\ x \log 8 &< \log\left(\frac{2}{3}\right) \\ x &< \frac{\log\left(\frac{2}{3}\right)}{\log 8} \end{aligned}$$

Note that the answer to Question **3** given in the coursebook uses the fact that

$$\begin{aligned} \log\left(\frac{2}{3}\right) &= -\log\left(\frac{2}{3}\right)^{-1} \\ &= -\log\left(\frac{3}{2}\right). \end{aligned}$$

4 $5^{x+3} = 7^{x-1}$

Taking logs to base 10 of both sides:

$$\begin{aligned} \log 5^{x+3} &= \log 7^{x-1} \\ (x+3)\log 5 &= (x-1)\log 7 \\ x \log 5 + 3 \log 5 &= x \log 7 - \log 7 \\ x(\log 7 - \log 5) &= 3 \log 5 + \log 7 \\ x &= \frac{3 \log 5 + \log 7}{\log 7 - \log 5} = 20.1 \end{aligned}$$

5
$$\begin{aligned} 6(4^x) - 11(2^x) + 4 &= 0 \\ 6 \times (2^2)^x - 11(2^x) + 4 &= 0 \\ 6(2^{2x}) - 11(2^x) + 4 &= 0 \\ 6(2^x)^2 - 11(2^x) + 4 &= 0 \end{aligned}$$

Let $u = 2^x$.

$$\begin{aligned} 6u^2 - 11u + 4 &= 0 \\ (3u-4)(2u-1) &= 0 \end{aligned}$$

$$u = \frac{4}{3} \qquad \text{or} \qquad u = \frac{1}{2}$$

$$2^x = \frac{4}{3} \qquad\qquad 2^x = \frac{1}{2}$$

$$\log 2^x = \log\left(\frac{4}{3}\right) \qquad 2^x = 2^{-1}$$

$$x \log 2 = \log\left(\frac{4}{3}\right) \qquad x = -1$$

$$x = \frac{\log\left(\frac{4}{3}\right)}{\log 2} = \frac{\log 2^2 - \log 3}{\log 2}$$

$$x = \frac{2\log 2 - \log 3}{\log 2}$$

The solutions are:

$x = \dfrac{2\log 2 - \log 3}{\log 2}$ or $x = -1$

6

$$\ln(5x+4) = 2\ln x + \ln 6$$
$$\ln(5x+4) = \ln x^2 + \ln 6$$
$$\ln(5x+4) = \ln(6x^2)$$
$$5x + 4 = 6x^2$$
$$6x^2 - 5x - 4 = 0$$
$$(3x-4)(2x+1) = 0$$
$$x = \frac{4}{3} \qquad \text{or} \qquad x = -\frac{1}{2}$$

But you cannot take the logarithm of a negative number, so $x > 0$.

$x = \dfrac{4}{3}$ is the only solution.

7 $y = Kx^m$

Taking natural logs of both sides:

$$\ln y = \ln K + \ln x^m$$
$$\ln y = m\ln x + \ln K$$
$$Y = mX + c$$
$$Y = \ln y$$
$$m = m$$
$$X = \ln x$$
$$c = \ln K$$

From the diagram:

$$\text{Gradient} = m = \frac{10.2 - 2.0}{6 - 0} = \frac{8.2}{6} = \frac{82}{60} = \frac{41}{30} = 1.366666\ \ldots\ldots = 1.37$$

$$y\text{–intercept} = 2 = c$$
$$\ln K = 2$$
$$K = e^2 = 7.39$$

8 **i**

$$y = 2^x$$
$$2^x + 3(2^{-x}) = 4$$
$$2^x + 3(2^x)^{-1} = 4$$
$$y + 3y^{-1} = 4$$
$$y + \frac{3}{y} - 4 = 0$$
$$y^2 - 4y + 3 = 0$$

ii Using the result from part **i**:

$$y^2 - 4y + 3 = 0$$
$$(y-3)(y-1) = 0$$

or

$$\begin{aligned} y &= 1 \\ 2^x &= 1 \\ x &= 0 \end{aligned} \qquad \begin{aligned} y &= 3 \\ 2^x &= 3 \\ \ln 2^x &= \ln 3 \\ x \ln 2 &= \ln 3 \\ x &= \frac{\ln 3}{\ln 2} = 1.58 \end{aligned}$$

The solutions are:

$x = 0$ or $x = 1.58$

9 $(1.2)^x = 6^y$

Taking natural logs of both sides:

$$\begin{aligned} \ln (1.2)^x &= \ln 6^y \\ x \ln 1.2 &= y \ln 6 \\ \frac{x}{y} &= \frac{\ln 6}{\ln 1.2} = 9.83 \end{aligned}$$

10 **i**

$$\begin{aligned} f(x) &= 12x^3 + 25x^2 - 4x - 12 \\ f(-2) &= 12(-2)^3 + 25(-2)^2 - 4(-2) - 12 \\ &= 12 \times (-8) + 25 \times 4 + 8 - 12 \\ &= -96 + 100 - 4 = 0 \end{aligned}$$

$(x + 2)$ is a factor of $f(x)$ by the factor theorem.

Dividing the cubic by $x + 2$ using long division:

$$\begin{array}{r} 12x^2 + x - 6 \\ x + 2 \overline{)12x^3 + 25x^2 - 4x - 12} \\ \underline{12x^3 + 24x^2} \qquad\qquad \\ x^2 - 4x \qquad \\ \underline{x^2 + 2x} \qquad \\ -6x - 12 \\ \underline{-6x - 12} \\ 0 \end{array}$$

$f(x) = (x + 2)(12x^2 + x - 6) = (x + 2)(3x - 2)(4x + 3)$

ii

$$\begin{aligned} 12 \times 27^y + 25 \times 9^y - 4 \times 3^y - 12 &= 0 \\ 12 \times (3^3)^y + 25 \times (3^2)^y - 4 \times 3^y - 12 &= 0 \\ 12(3^y)^3 + 25(3^y)^2 - 4(3^y) - 12 &= 0 \end{aligned}$$

3^y is a root of $f(x) = 0$.

From part **i**, $f(x) = 0$ if $(x + 2)(3x - 2)(4x + 3) = 0$.

$x = -2$ or $x = \frac{2}{3}$ or $x = -\frac{3}{4}$

So

$3^y = -2$ or $3^y = \frac{2}{3}$ or $3^y = -\frac{3}{4}$

But $3^y > 0$ for all values of y.

So $3^y = \frac{2}{3}$ is the only possibility.

Taking logs to base 10 of both sides:

$$\begin{aligned} \log 3^y &= \log \frac{2}{3} \\ y \log 3 &= \log \frac{2}{3} \\ y &= \frac{\log \frac{2}{3}}{\log 3} = -0.369 \end{aligned}$$

11

$$\begin{aligned} |4 - 2^x| &= 10 \\ 4 - 2^x &= 10 \\ 2^x &= -6 \end{aligned}$$

No real solution because $2^x > 0$.

Or

$$\begin{aligned} 4 - 2^x &= -10 \\ 2^x &= 14 \end{aligned}$$

Taking logs to base 10 of both sides:

$$\begin{aligned} \log 2^x &= \log 14 \\ x \log 2 &= \log 14 \\ x &= \frac{\log 14}{\log 2} = 3.81 \end{aligned}$$

12 $\mathrm{e}^x = 3^{x-2}$

Taking natural logs of both sides:

$$\begin{aligned} \ln e^x &= \ln 3^{x-2} \\ x &= (x-2)\ln 3 \\ x &= x\ln 3 - 2\ln 3 \\ x\ln 3 - x &= 2\ln 3 \\ x(\ln 3 - 1) &= 2\ln 3 \\ x &= \frac{2\ln 3}{\ln 3 - 1} = 22.281 \end{aligned}$$

13

$$\begin{aligned} 3^x + 3^{2x} &= 3^{3x} \\ 3^x + (3^x)^2 &= (3^x)^3 \end{aligned}$$

Let $u = 3^x$.

$$\begin{aligned} u + u^2 &= u^3 \\ u^3 - u^2 - u &= 0 \\ u(u^2 - u - 1) &= 0 \end{aligned}$$

$u = 0$ or $u^2 - u - 1 = 0$

$u = 0$ or $u = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2 \times 1} = \dfrac{1 \pm \sqrt{5}}{2}$

$3^x = 0$ or $\dfrac{1+\sqrt{5}}{2}$ or $\dfrac{1-\sqrt{5}}{2}$

$3^x > 0$, so the only solution is:

$3^x = \dfrac{1+\sqrt{5}}{2}$

Taking logs to base 10 of both sides:

$$\begin{aligned} \log 3^x &= \log\left(\frac{1+\sqrt{5}}{2}\right) \\ x\log 3 &= \log\left(\frac{1+\sqrt{5}}{2}\right) \\ x &= \frac{\log\left(\frac{1+\sqrt{5}}{2}\right)}{\log 3} = 0.438 \end{aligned}$$

14 **a** $5^y = 3^{2x-4}$

Taking natural logs of both sides:

$$\ln 5^y = \ln 3^{2x-4}$$
$$y\ln 5 = (2x-4)\ln 3$$
$$y\ln 5 = 2x\ln 3 - 4\ln 3$$
$$y = \left(\frac{2\ln 3}{\ln 5}\right)x - \frac{4\ln 3}{\ln 5}$$
$$Y = mX + c$$
$$Y = y$$
$$m = \frac{2\ln 3}{\ln 5}$$
$$X = x$$
$$c = -\frac{4\ln 3}{\ln 5}$$
$$\text{gradient} = \frac{2\ln 3}{\ln 5}$$

b $y = \left(\frac{2\ln 3}{\ln 5}\right)x - \frac{4\ln 3}{\ln 5}$

When $x = 0$:

$$y = -\frac{4\ln 3}{\ln 5}$$

When $y = 0$:

$$\left(\frac{2\ln 3}{\ln 5}\right)x = \frac{4\ln 3}{\ln 5}$$
$$x = 2$$

Point P is $(2,\ 0)$.

Point Q is $\left(0,\ -\frac{4\ln 3}{\ln 5}\right)$.

Midpoint is $\left(\frac{2+0}{2}, \frac{0 + -\frac{4\ln 3}{\ln 5}}{2}\right)$.

So the midpoint is at $\left(1, -\frac{2\ln 3}{\ln 5}\right)$.

15 $y = K(b^x)$

Taking natural logs of both sides:

$$\ln y = \ln K + \ln b^x$$
$$\ln y = (\ln b)x + \ln K$$
$$Y = mX + c$$
$$Y = \ln y$$
$$m = \ln b$$
$$X = x$$
$$c = \ln K$$

From the diagram:

$$\text{Gradient} = m = \frac{2.1 - 1.7}{3.1 - 2.3} = \frac{0.4}{0.8} = \frac{1}{2}$$
$$\ln b = \frac{1}{2}$$
$$b = e^{\frac{1}{2}} = 1.65$$

Line passes through $(2.3,\ 1.7)$:

$$Y = mX + c$$
$$1.7 = \frac{1}{2}(2.3) + c$$
$$c = 1.7 - 1.15 = 0.55$$
$$\ln K = 0.55$$
$$K = e^{0.55} = 1.73$$

Chapter 3

Trigonometry

EXERCISE 3A

You should learn the standard trigonometric ratios for 0, 30, 45, 60 and 90°. You will be able to use these more quickly, when needed, if you know them well.

1 **b** $\text{cosec } 45^\circ$

$= \frac{1}{\sin 45^\circ}$

$= \frac{1}{\left(\frac{\sqrt{2}}{2}\right)}$

$= \frac{2}{\sqrt{2}}$

$= \sqrt{2}$

f $\cot 330^\circ$

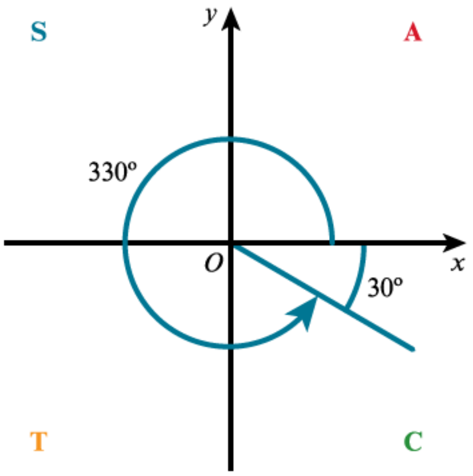
$= \frac{1}{\tan 330^\circ}$

$= -\frac{1}{\tan 30^\circ}$

$= -\frac{1}{\left(\frac{\sqrt{3}}{3}\right)}$

$= -\frac{3}{\sqrt{3}}$

$= -\sqrt{3}$



–

2 **c** $\sec\frac{\pi}{4}$

$=\dfrac{1}{\cos\frac{\pi}{4}}$

$=\dfrac{1}{\left(\frac{\sqrt{2}}{2}\right)}$

$=\dfrac{2}{\sqrt{2}}$

$=\sqrt{2}$

g $\cot\frac{4\pi}{3}$

$=\dfrac{1}{\tan\frac{4\pi}{3}}$

$=\dfrac{1}{\tan\frac{\pi}{3}}$

$=\dfrac{1}{\sqrt{3}}$

> Sometimes your calculation will end in a surd that looks different to the answer in the coursebook. Try rationalising the denominator. This will help you to check that your answer is, in fact, the same as the one shown in the coursebook.

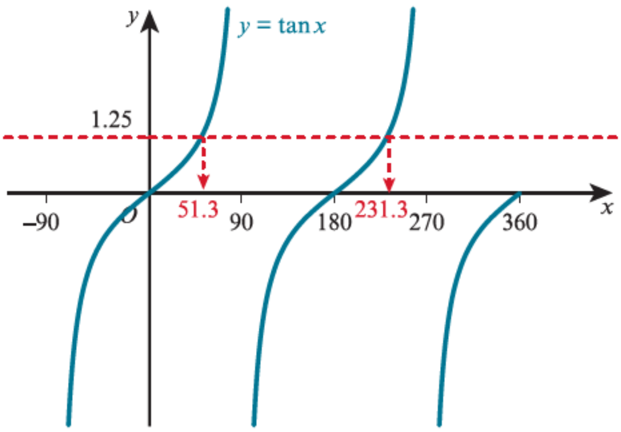
3 **b** $\cot x = 0.8$

$\dfrac{1}{\tan x} = 0.8$

$\tan x = \dfrac{1}{0.8} = 1.25$

$x = 51.3^\circ$ (calculator)

or $x = 51.3^\circ + 180^\circ = 231.3^\circ$



> You should always show how you arrived at an answer, either by drawing an appropriate quadrants (ASTC) diagram or a graph.

d $3\sec x - 4 = 0$

$3\sec x = 4$

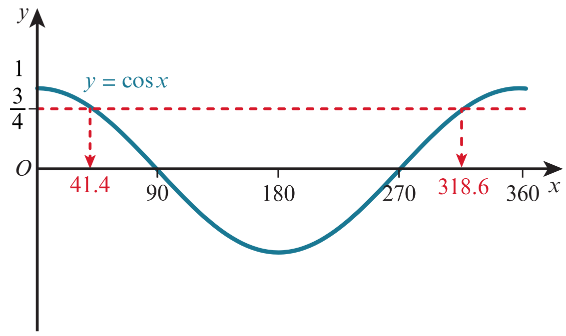
$\sec x = \dfrac{4}{3}$

$\dfrac{1}{\cos x} = \dfrac{4}{3}$

$\cos x = \dfrac{3}{4}$

$x = 41.4^\circ$ (calculator)

or $x = 360^\circ - 41.4^\circ = 318.6^\circ$



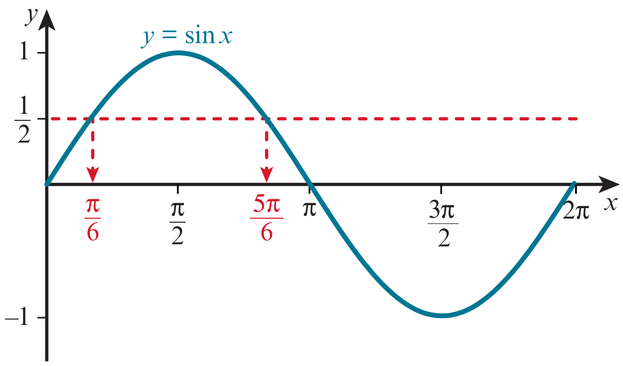
4 **a** $\operatorname{cosec} x = 2$

$$\frac{1}{\sin x} = 2$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} \text{ (calculator)}$$

$$\text{or } x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$



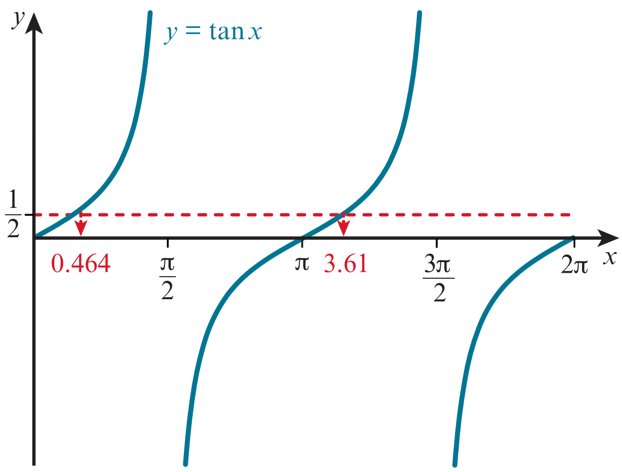
c $\cot x = 2$

$$\frac{1}{\tan x} = 2$$

$$\tan x = \frac{1}{2}$$

$$x = 0.464$$

$$\text{or } x = 0.464 + \pi = 3.61$$



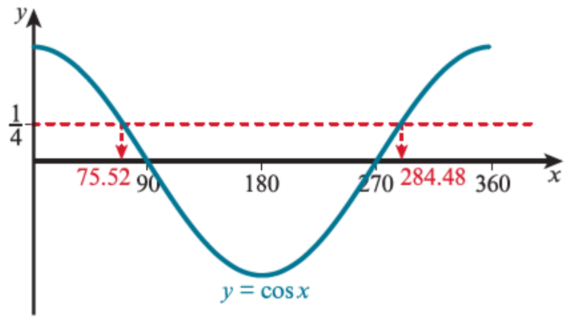
5 **b** $\sec 2x = 4$

$$\frac{1}{\cos 2x} = 4$$

$$\cos 2x = \frac{1}{4}$$

$$2x = 75.522\ldots^\circ \text{ or } 2x = 360^\circ - 75.522\ldots^\circ = 284.477\ldots^\circ$$

$$x = 37.8^\circ \text{ or } x = 142.2^\circ$$



In each question on this page there is a moment when the trigonometric function disappears and several angles are given. There are multiple solutions at this point, each time, because the graphs of those functions repeat. In 5b you will see that all of the necessary solutions have been found before we divide by 2.

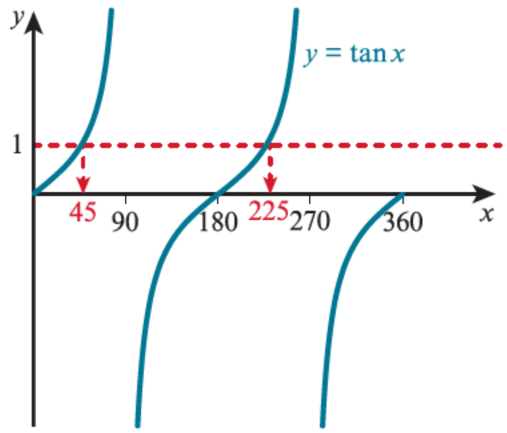
c $\cot 2x = 1$

$\frac{1}{\tan 2x} = 1$

$\tan 2x = 1$

$2x = 45^\circ$ (calculator) or $2x = 45^\circ + 180^\circ = 225^\circ$

$x = 22.5^\circ$ or $x = 112.5^\circ$



6 **b** $\sec(2x + 60^\circ) = -1.5$

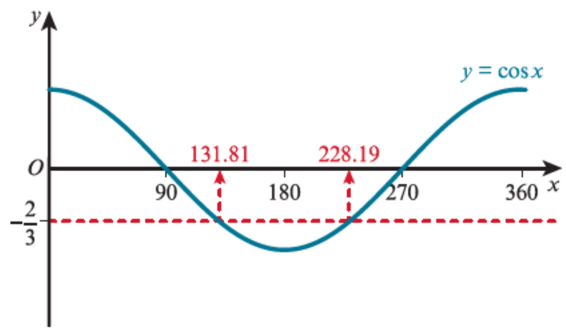
$\frac{1}{\cos(2x + 60^\circ)} = -\frac{3}{2}$

$\cos(2x + 60^\circ) = -\frac{2}{3}$

$2x + 60^\circ = 131.810\ldots^\circ$ or $2x + 60^\circ = 360^\circ - 131.810\ldots^\circ = 228.190\ldots^\circ$

$2x = 71.810\ldots^\circ$ or $2x = 168.189\ldots^\circ$

$x = 35.9^\circ$ or $x = 84.1^\circ$



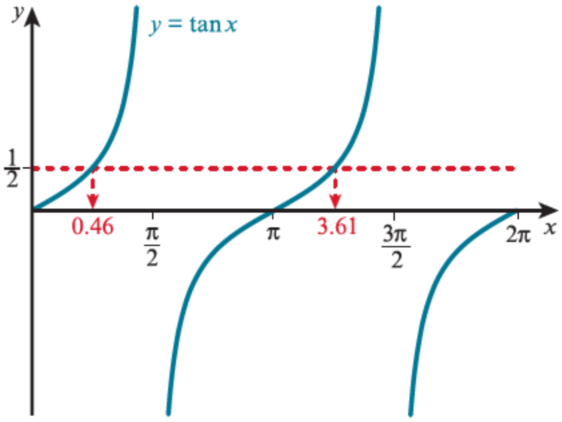
c $\cot\left(x+\frac{\pi}{4}\right)=2$

$$\frac{1}{\tan\left(x+\frac{\pi}{4}\right)}=2$$

$$\tan\left(x+\frac{\pi}{4}\right)=\frac{1}{2}$$

$x+\frac{\pi}{4}=0.463647\ldots$ or $x+\frac{\pi}{4}=0.463647\ldots+\pi=3.6052\ldots$

$x=2.82$ or $x=5.96$



7 **a** $\operatorname{cosec}^2 x=4$

$$\frac{1}{\sin^2 x}=4$$

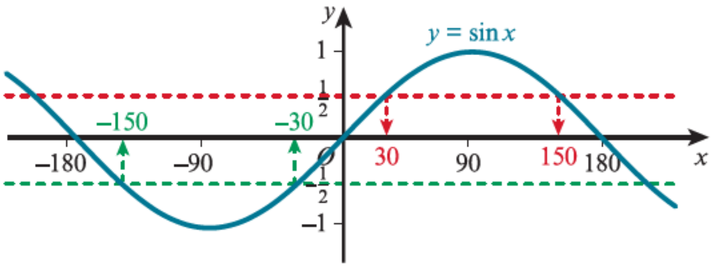
$$\sin^2 x=\frac{1}{4}$$

$\sin x=\frac{1}{2}$ or $\sin x=-\frac{1}{2}$

$x=30^\circ$ $x=-30^\circ$

or or

$x=180^\circ-30^\circ=150^\circ$ $x=180^\circ-(-30^\circ)-360^\circ=-150^\circ$



Note that the symmetry of the graph is the most important thing. Remember that the graphs repeat every 360° (for sine and cosine) or 180° (for tangent). If you find that you have a solution that is out of range, you can make use of this repeating pattern.

e $\operatorname{cosec} x=\sec x$

$$\frac{1}{\sin x}=\frac{1}{\cos x}$$

$$\frac{\sin x}{\cos x}=1$$

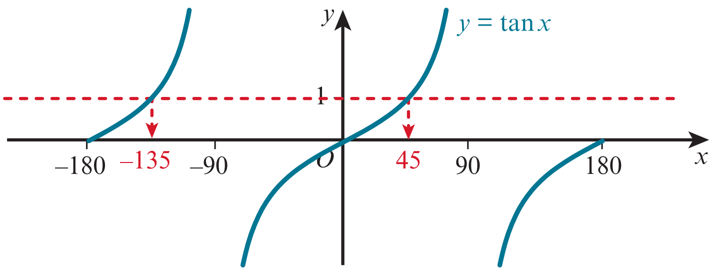
$$\tan x=1$$

$x=45^\circ$ or $45^\circ+180^\circ=225^\circ$

225° is out of the range, so use $225^\circ-360^\circ=-135^\circ$

So

$x=45^\circ$ or $x=-135^\circ$ only



8 **e** $\tan^2\theta + 3\sec\theta = 0$

Now, using the fact that $1 + \tan^2\theta = \sec^2\theta$:

$\sec^2\theta - 1 + 3\sec\theta = 0$

$\sec^2\theta + 3\sec\theta - 1 = 0$

Letting $u = \sec\theta$:

$u^2 + 3u - 1 = 0$

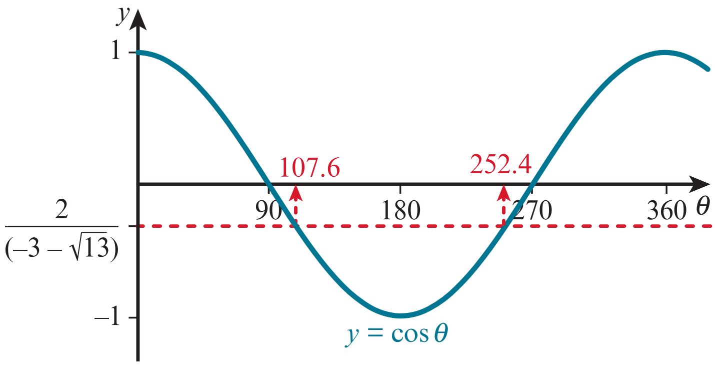
$$u = \frac{-3 \pm \sqrt{3^2 - 4\times 1\times(-1)}}{2\times 1} = \frac{-3\pm\sqrt{13}}{2}$$

$$\sec\theta = \frac{-3+\sqrt{13}}{2} \text{ or } \sec\theta = \frac{-3-\sqrt{13}}{2}$$

$$\cos\theta = \frac{2}{-3+\sqrt{13}} \text{ or } \cos\theta = \frac{2}{-3-\sqrt{13}}$$

$\cos\theta = \dfrac{2}{-3+\sqrt{13}}$ has no solutions since $\dfrac{2}{-3+\sqrt{13}} > 1.$

$\cos\theta = \dfrac{2}{-3-\sqrt{13}}$ has solutions $\theta = 107.6^\circ$ or $\theta = 360^\circ - 107.6^\circ = 252.4^\circ$.



f

$$\sqrt{3}\sec^2\theta = \operatorname{cosec}\theta$$

$$\frac{\sqrt{3}}{\cos^2\theta} = \frac{1}{\sin\theta}$$

$$\sqrt{3}\sin\theta = \cos^2\theta$$

$$\sqrt{3}\sin\theta = 1 - \sin^2\theta$$

$\sin^2\theta + \sqrt{3}\sin\theta - 1 = 0$

Letting $u = \sin\theta$:

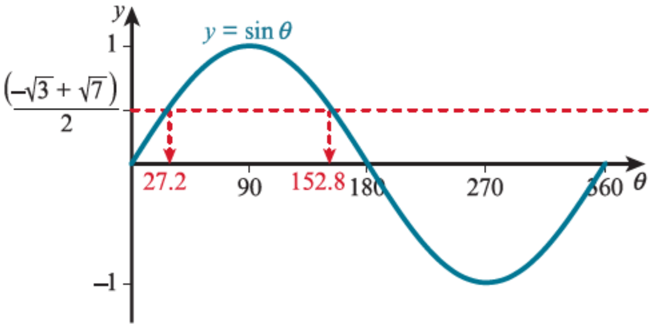
$u^2 + \sqrt{3}u - 1 = 0$

$$u = \frac{-\sqrt{3}\pm\sqrt{3 - 4\times 1\times(-1)}}{2\times 1} = \frac{-\sqrt{3}\pm\sqrt{7}}{2}$$

$\sin\theta = \dfrac{-\sqrt{3}-\sqrt{7}}{2}$ has no solutions since $\dfrac{-\sqrt{3}-\sqrt{7}}{2} < -1.$

$\sin\theta = \dfrac{-\sqrt{3}+\sqrt{7}}{2}$ has solutions $\theta = 27.2^\circ$ (calculator).

or $\theta = 180^\circ - 27.2^\circ = 152.8^\circ$



When questions contain more than one trigonometric function, you can often find ways of rewriting one or more terms to get something more useful.

9 **a**

$$\sec\theta = 3\cos\theta - \tan\theta$$

$$\frac{1}{\cos\theta} = 3\cos\theta - \frac{\sin\theta}{\cos\theta}$$

$$3\cos^2\theta - \sin\theta = 1$$

Using the fact that $\cos^2\theta + \sin^2\theta = 1$:

$$3\left(1 - \sin^2\theta\right) - \sin\theta - 1 = 0$$

$$3 - 3\sin^2\theta - \sin\theta - 1 = 0$$

$$3\sin^2\theta + \sin\theta - 2 = 0$$

Letting $u = \sin\theta$:

$$3u^2 + u - 2 = 0$$

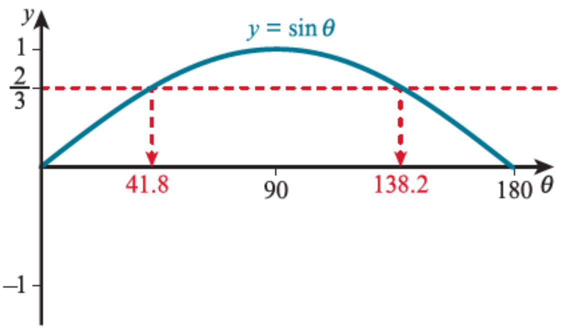
$$(3u - 2)(u + 1) = 0$$

$$u = \frac{2}{3} \text{ or } u = -1$$

$$\sin\theta = \frac{2}{3} \text{ or } \sin\theta = -1$$

$\theta = 41.8^\circ$ or $\theta = -90^\circ, 270^\circ$ (both out of range)

or $\theta = 180^\circ - 41.8^\circ = 138.2^\circ$



For many questions like this it is useful to keep more decimal places than required until you reach your final answer(s). If you round too soon, you might find that your answers are slightly inaccurate.

d $2\cot^2\theta + 7\operatorname{cosec}2\theta = 2$

Using $1 + \cot^2\theta = \operatorname{cosec}^2\theta$:

$$2\left(\operatorname{cosec}^2\theta - 1\right) + 7\operatorname{cosec}2\theta = 2$$

$$2\operatorname{cosec}^2 2\theta - 2 + 7\operatorname{cosec}2\theta - 2 = 0$$

$$2\operatorname{cosec}^2 2\theta + 7\operatorname{cosec}2\theta - 4 = 0$$

Letting $u = \operatorname{cosec}2\theta$:

$$2u^2 + 7u - 4 = 0$$

$$(2u - 1)(u + 4) = 0$$

$$u = \frac{1}{2} \text{ or } u = -4$$

$\operatorname{cosec} 2\theta = \frac{1}{2}$ or $\operatorname{cosec} 2\theta = -4$

$\frac{1}{\sin 2\theta} = \frac{1}{2}$ or $\frac{1}{\sin 2\theta} = -4$

$\sin 2\theta = 2$ or $\sin 2\theta = -\frac{1}{4}$

$\sin 2\theta = 2$ has no solutions.

$\sin 2\theta = -\frac{1}{4}$ has solutions $2\theta = -14.4775\ .....\ ^\circ$ (calculator).

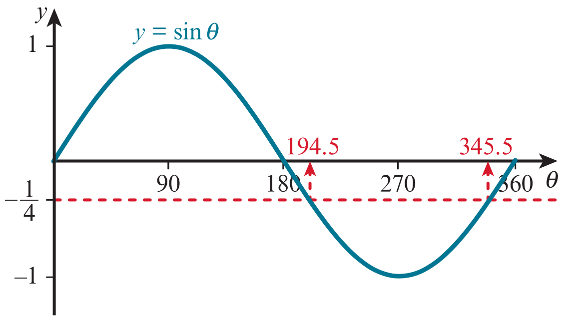
or $2\theta = 180^\circ - (-14.4775\ .....\ ^\circ) = 194.4775\ .....\ ^\circ$

or $2\theta = -14.4775\ .....\ ^\circ + 360^\circ = 345.522\ .....\ ^\circ$

$\theta = -7.3^\circ$ (out of range)

$\theta = 97.2^\circ$

$\theta = 172.8^\circ$



10 **a** $\tan^2\theta + 3\sec\theta + 3 = 0$

Using $\tan^2\theta = \sec^2\theta - 1$:

$\sec^2\theta - 1 + 3\sec\theta + 3 = 0$

$\sec^2\theta + 3\sec\theta + 2 = 0$

Letting $u = \sec\theta$:

$u^2 + 3u + 2 = 0$

$(u+1)(u+2) = 0$

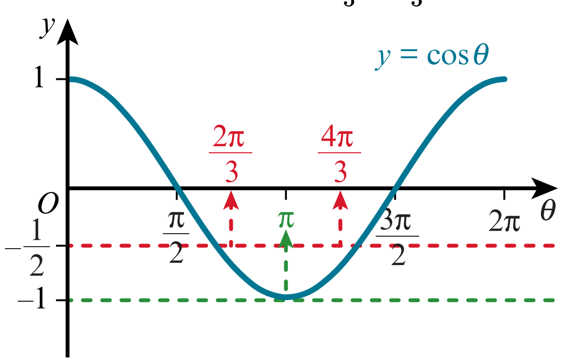
$u = -1$ or $u = -2$

$\sec\theta = -1$ or $\sec\theta = -2$

$\cos\theta = -1$ or $\cos\theta = -\frac{1}{2}$

$\theta = \pi$ (calculator) or $\theta = \frac{2\pi}{3}$

or $\theta = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$



b $3\cot^2\theta + 5\operatorname{cosec}\theta + 1 = 0$

Using $\cot^2\theta = \operatorname{cosec}^2\theta - 1$:

$3\left(\operatorname{cosec}^2\theta - 1\right) + 5\ \operatorname{cosec}\theta + 1 = 0$

$3\operatorname{cosec}^2\theta - 3 + 5\ \operatorname{cosec}\theta + 1 = 0$

$3\operatorname{cosec}^2\theta + 5\ \operatorname{cosec}\theta - 2 = 0$

Letting $u = \operatorname{cosec}\theta$:

$$3u^2 + 5u - 2 = 0$$
$$(3u - 1)(u + 2) = 0$$
$u = -2$ or $u = \frac{1}{3}$

$\operatorname{cosec}\theta = -2$ or $\operatorname{cosec}\theta = \frac{1}{3}$

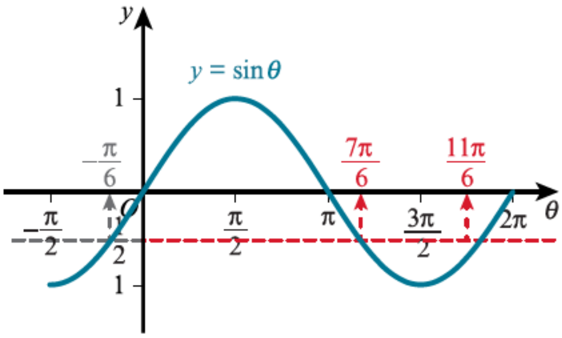
$\sin\theta = -\frac{1}{2}$ or $\sin\theta = 3$ (no solutions)

$\theta = -\frac{\pi}{6}$ (calculator)

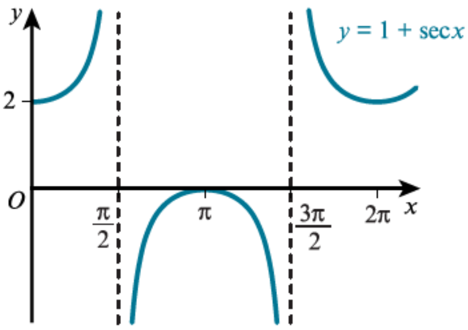
or $\theta = \pi - \left(-\frac{\pi}{6}\right) = \frac{7\pi}{6}$

or $\theta = -\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}$

Only the last two solutions are in the required range, although the first solution is used to find the other two.

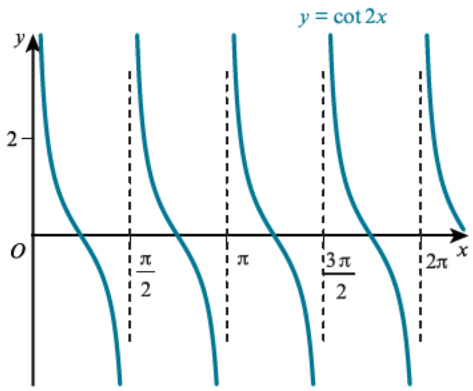


11 **a, b i** This is the graph of $y = \sec x$ translated with column vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



The asymptotes lie at the same places as for $y = \sec x$, i.e. at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

ii This is the graph of $y = \cot x$, stretched factor $\frac{1}{2}$ parallel to the x-axis.



The asymptotes lie at $x = \frac{\pi}{2}, x = \pi, x = \frac{3\pi}{2}, x = 2\pi$

In the solutions to Question **12** it is common to take the more complicated side of the identity first. You must avoid manipulating BOTH sides of the identity at the same time.

12 **a** Taking the left-hand side:

$\sin x + \cos x \cot x$

$\equiv \sin x + \cos x \times \dfrac{\cos x}{\sin x}$

$\equiv \sin x + \dfrac{\cos^2 x}{\sin x}$

$\equiv \dfrac{\sin^2 x + \cos^2 x}{\sin x}$

$\equiv \dfrac{1}{\sin x}$

$\equiv \operatorname{cosec} x$

$\equiv$ Right-hand side (as required)

d Taking the left-hand side:

$(1 + \sec x)(\operatorname{cosec} x - \cot x)$

$\equiv \operatorname{cosec} x - \cot x + \sec x \operatorname{cosec} x - \sec x \cot x$

$\equiv \dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x} + \dfrac{1}{\cos x \sin x} - \dfrac{\cos x}{\cos x \sin x}$

$\equiv -\dfrac{\cos x}{\sin x} + \dfrac{1}{\cos x \sin x}$

$\equiv -\dfrac{\cos^2 x}{\sin x \cos x} + \dfrac{1}{\cos x \sin x}$

$\equiv \dfrac{1 - \cos^2 x}{\sin x \cos x}$

$\equiv \dfrac{\sin^2 x}{\sin x \cos x}$

$\equiv \dfrac{\sin x}{\cos x}$

$\equiv \tan x$

$\equiv$ Right-hand side (as required)

13 **c** Taking the left-hand side:

$\equiv \dfrac{1 - \cos^2 x}{\sec^2 x - 1}$

$\equiv \dfrac{\sin^2 x}{\tan^2 x}$

$\equiv \dfrac{\sin^2 x}{\left(\dfrac{\sin^2 x}{\cos^2 x}\right)}$

$\equiv \dfrac{\sin^2 x \cos^2 x}{\sin^2 x}$

$\equiv \cos^2 x$

$\equiv 1 - \sin^2 x$

$\equiv$ Right-hand side (as required)

g Taking the left-hand side:

$\dfrac{1}{1 + \cos x} + \dfrac{1}{1 - \cos x}$

$\equiv \dfrac{1 - \cos x}{(1 + \cos x)(1 - \cos x)} + \dfrac{1 + \cos x}{(1 + \cos x)(1 - \cos x)}$

$\equiv \dfrac{1 - \cos x + 1 + \cos x}{1 - \cos^2 x}$

$\equiv \dfrac{2}{\sin^2 x}$

$\equiv 2 \operatorname{cosec}^2 x$

$\equiv$ Right-hand side (as required)

14 **a** $6 \sec^3\theta - 5 \sec^2\theta - 8 \sec\theta + 3 = 0$

Let $u = \sec\theta$.

$$6u^3 - 5u^2 - 8u + 3 = 0$$
$$6(-1)^3 - 5(-1)^2 - 8(-1) + 3 = 0$$

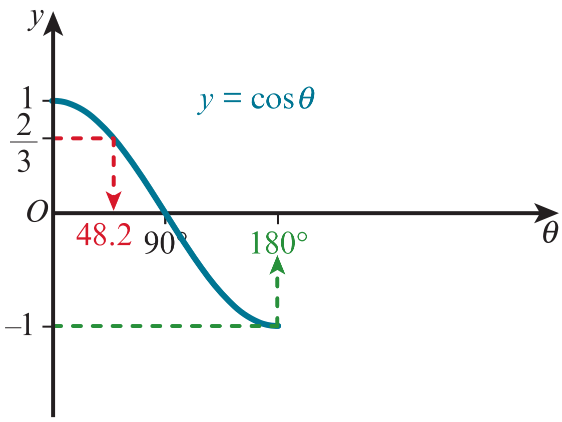
$u + 1$ is a factor of $6u^3 - 5u^2 - 8u + 3$ by the factor theorem.

Dividing:

$$\begin{array}{r} 6u^2 - 11u + 3 \\ u+1 \overline{)\, 6u^3 - 5u^2 - 8u + 3} \\ \underline{6u^3 + 6u^2} \\ -11u^2 - 8u \\ \underline{-11u^2 - 11u} \\ 3u + 3 \\ \underline{3u + 3} \\ 0 \end{array}$$

$$(u+1)\left(6u^2 - 11u + 3\right) = 0$$
$$(u+1)(3u-1)(2u-3) = 0.$$

$u = -1$	$u = \frac{1}{3}$	$u = \frac{3}{2}$
$\sec\theta = -1$	$\sec\theta = \frac{1}{3}$	$\sec\theta = \frac{3}{2}$
$\cos\theta = -1$	$\cos\theta = 3$	$\cos\theta = \frac{2}{3}$
$\theta = 180^\circ$	No solutions	$\theta = 48.2^\circ$



b $$2\cot^3\theta + 3\operatorname{cosec}^2\theta - 8\cot\theta = 0$$
$$2\cot^3\theta + 3(1 + \cot^2\theta) - 8\cot\theta = 0$$
$$2\cot^3\theta + 3 + 3\cot^2\theta - 8\cot\theta = 0$$
$$2\cot^3\theta + 3\cot^2\theta - 8\cot\theta + 3 = 0$$

Let $u = \cot\theta$.

$$2u^3 + 3u^2 - 8u + 3 = 0$$
$$2(1)^3 + 3(1)^2 - 8(1) + 3 = 0$$

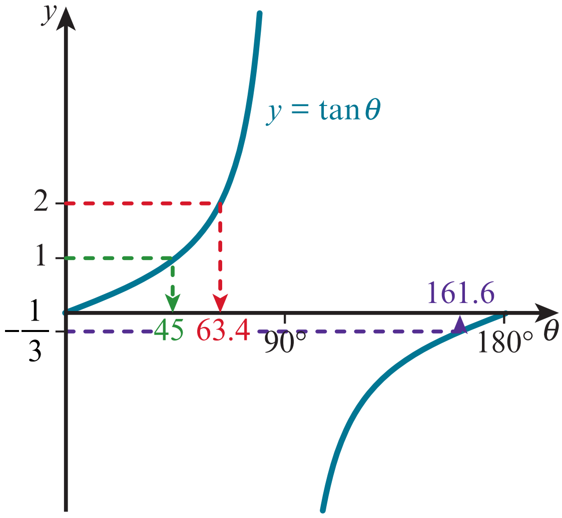
$u - 1$ is a factor of $2u^3 + 3u^2 - 8u + 3$ by the factor theorem.

Dividing:

$$\begin{array}{r} 2u^2 + 5u - 3 \\ u-1 \overline{)\, 2u^3 + 3u^2 - 8u + 3} \\ \underline{2u^3 - 2u^2} \\ 5u^2 - 8u \\ \underline{5u^2 - 5u} \\ -3u + 3 \\ \underline{-3u + 3} \\ 0 \end{array}$$

$$(u-1)\left(2u^2 + 5u - 3\right) = 0$$
$$(u-1)(2u-1)(u+3) = 0$$

$u = 1$	$u = \frac{1}{2}$	$u = -3$
$\cot\theta = 1$	$\cot\theta = \frac{1}{2}$	$\cot\theta = -3$
$\tan\theta = 1$	$\tan\theta = 2$	$\tan\theta = -\frac{1}{3}$
$\theta = 45^\circ$	$\theta = 63.4^\circ$	$\theta = -18.4^\circ + 180 = 161.6^\circ$



EXERCISE 3B

Questions such as those in this exercise are often easier if you have memorised the formula.

1 Using $\cos(A+B) = \cos A \cos B - \sin A \sin B$:

$$\begin{aligned}&\cos(x+30^\circ)\\&= \cos x \cos 30^\circ - \sin x \sin 30^\circ\\&= \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\end{aligned}$$

2 **a** Using $\sin(A+B) = \sin A \cos B + \cos A \sin B$:

$$\begin{aligned}&\sin 20^\circ \cos 70^\circ + \cos 20^\circ \sin 70^\circ\\&= \sin(20^\circ + 70^\circ)\\&= \sin 90^\circ\\&= 1\end{aligned}$$

e Using $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$:

$$\begin{aligned}&\frac{\tan 25^\circ + \tan 20^\circ}{1 - \tan 25^\circ \tan 20^\circ}\\&= \tan(25^\circ + 20^\circ)\\&= \tan 45^\circ\\&= 1\end{aligned}$$

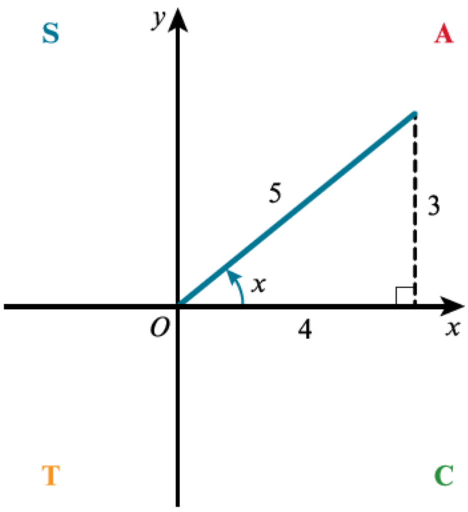
3 **c** $\cos 105^\circ$

$$\begin{aligned}&= \cos(60^\circ + 45^\circ)\\&= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ\\&= \frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}\\&= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\\&= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

d $\tan(-15^\circ)$

$$\begin{aligned}&= \tan(30^\circ - 45^\circ)\\&= \frac{\tan 30^\circ - \tan 45^\circ}{1 + \tan 30^\circ \tan 45^\circ}\\&= \frac{\frac{1}{\sqrt{3}} - 1}{1 + \frac{1}{\sqrt{3}} \times 1}\\&= \frac{1 - \sqrt{3}}{\sqrt{3} + 1}\\&= \frac{(1 - \sqrt{3})^2}{(\sqrt{3} + 1)(1 - \sqrt{3})}\\&= \frac{1 - 2\sqrt{3} + 3}{1 - 3}\\&= \frac{4 - 2\sqrt{3}}{-2}\\&= -2 + \sqrt{3}\end{aligned}$$

4 $\cos x = \dfrac{4}{5}$



$\sin x = \frac{3}{5}$

$\cos(x - 60^{\circ})$

$= \cos x \cos 60^{\circ} + \sin x \sin 60^{\circ}$

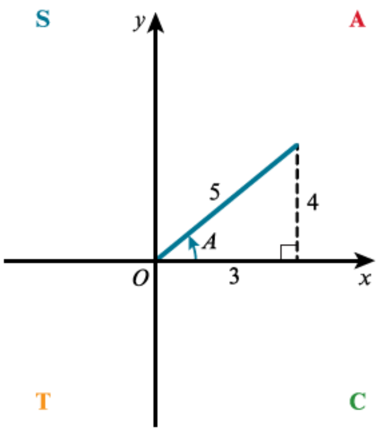
$= \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x$

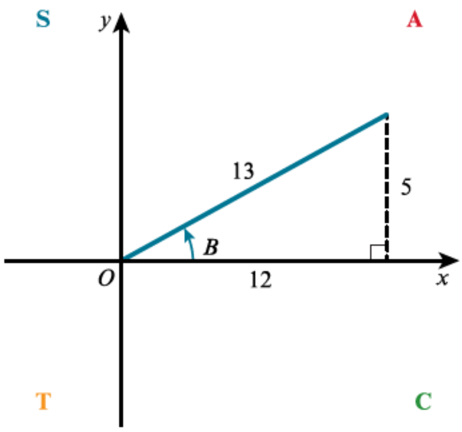
$= \left(\frac{1}{2}\right)\left(\frac{4}{5}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{3}{5}\right)$

$= \frac{4 + 3\sqrt{3}}{10}$

Use brackets next to each other to indicate multiplication, rather than the multiplication symbol ×. This will then help you to avoid confusion.

5 $\sin A = \frac{4}{5}$ $\sin B = \frac{5}{13}$

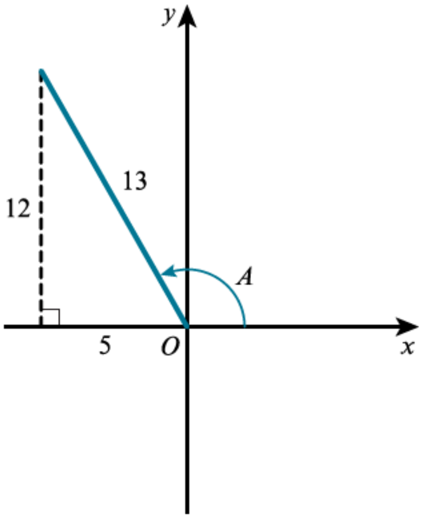


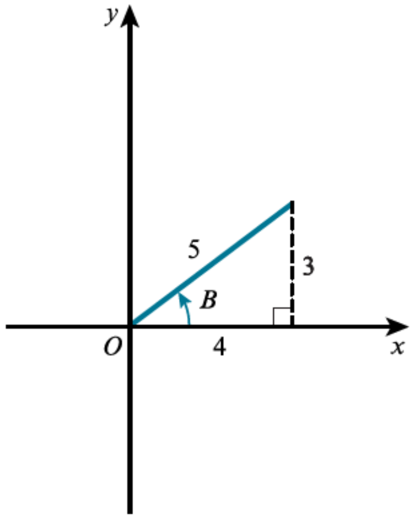


$\cos A = \frac{3}{5}$ $\cos B = \frac{12}{13}$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$= \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{5}{13}\right)$$
$$= \frac{48 + 15}{5 \times 13}$$
$$= \frac{63}{65}$$

6 $\sin A = \frac{12}{13}$ $\sin B = \frac{3}{5}$





$\cos A = -\frac{5}{13}$ $\cos B = \frac{4}{5}$

$\tan A = -\frac{12}{5}$ $\tan B = \frac{3}{4}$

a $\sin(A + B)$

$= \sin A \cos B + \cos A \sin B$

$= \left(\frac{12}{13}\right)\left(\frac{4}{5}\right) + \left(-\frac{5}{13}\right)\left(\frac{3}{5}\right)$

$= \frac{48 - 15}{65}$

$= \frac{33}{65}$

b $\cos(A - B)$

$= \cos A \cos B + \sin A \sin B$

$= \left(-\frac{5}{13}\right)\left(\frac{4}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{3}{5}\right)$

$= \frac{-20 + 36}{65}$

$= \frac{16}{65}$

c $\tan(A + B)$

$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$

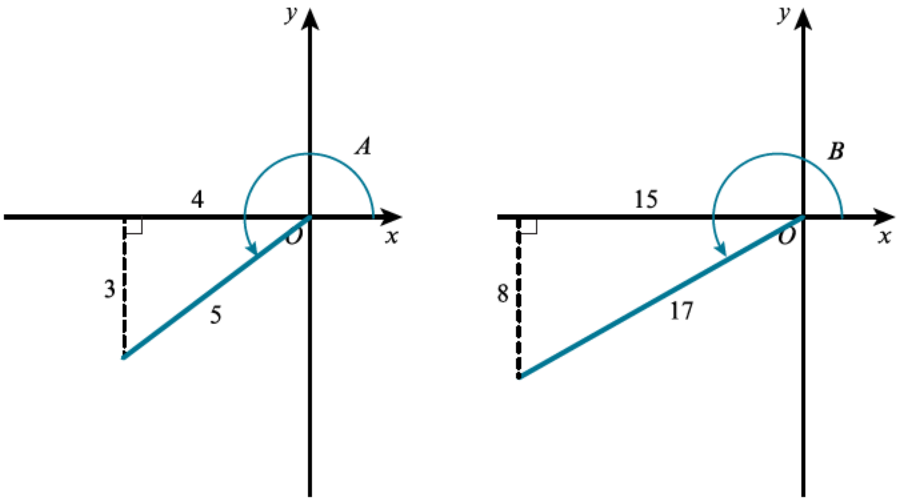
$= \frac{-\frac{12}{5} + \frac{3}{4}}{1 - \left(-\frac{12}{5}\right)\left(\frac{3}{4}\right)}$

$= \frac{\left(\frac{-48 + 15}{20}\right)}{1 - \left(-\frac{36}{20}\right)}$

$= \frac{\left(-\frac{33}{20}\right)}{\left(\frac{56}{20}\right)}$

$= -\frac{33}{56}$

7 Both A and B are in the same quadrant and their respective cosine and sines are both negative. A and B must both lie in the third quadrant:



For questions with multiple parts (like Question **7** here), it is always good to work out of all of the standard trigonometric ratios first.

$$\sin A = -\frac{3}{5} \qquad \cos A = -\frac{4}{5} \qquad \tan A = \frac{3}{4}$$

$$\sin B = -\frac{8}{17} \qquad \cos B = -\frac{15}{17} \qquad \tan B = \frac{8}{15}$$

a $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$= \left(-\frac{3}{5}\right)\left(-\frac{15}{17}\right) + \left(-\frac{4}{5}\right)\left(-\frac{8}{17}\right)$$

$$= \frac{45+32}{85}$$

$$= \frac{77}{85}$$

b $\cos(A+B)$

$$= \cos A \cos B - \sin A \sin B$$

$$= \left(-\frac{4}{5}\right)\left(-\frac{15}{17}\right) - \left(-\frac{3}{5}\right)\left(-\frac{8}{17}\right)$$

$$= \frac{60-24}{85}$$

$$= \frac{36}{85}$$

c $\tan(A-B)$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$= \frac{\frac{3}{4} - \frac{8}{15}}{1 + \left(\frac{3}{4}\right)\left(\frac{8}{15}\right)}$$

$$= \frac{\left(\frac{45-32}{60}\right)}{1 + \left(\frac{24}{60}\right)}$$

$$= \frac{\left(\frac{13}{60}\right)}{\left(\frac{84}{60}\right)}$$

$$= \frac{13}{84}$$

8 $\tan(A-B) = 2$

$$\frac{\tan A - \tan B}{1 + \tan A \tan B} = 2$$

Substituting $\tan A = t$:

$$\frac{t - \tan B}{1 + t\tan B} = 2$$
$$t - \tan B = 2 + 2t\tan B$$
$$2t\tan B + \tan B = t - 2$$
$$(2t + 1)\tan B = t - 2$$
$$\tan B = \frac{t-2}{2t+1}$$

9

$$\cos(A - B) = 3\cos(A + B)$$
$$\cos A\cos B + \sin A\sin B = 3\cos A\cos B - 3\sin A\sin B$$
$$2\cos A\cos B = 4\sin A\sin B$$
$$\frac{\sin A\sin B}{\cos A\cos B} = \frac{2}{4}$$
$$\tan A\tan B = \frac{1}{2}$$

10 **a**

$$8 + \operatorname{cosec}^2\theta = 6\cot\theta$$

Using $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$:

$$8 + \left(1 + \cot^2\theta\right) = 6\cot\theta$$
$$\cot^2\theta - 6\cot\theta + 9 = 0$$

Letting $u = \cot\theta$:

$$u^2 - 6u + 9 = 0$$
$$(u-3)^2 = 0$$
$$u = 3$$
$$\cot\theta = 3$$
$$\tan\theta = \frac{1}{3}$$

b $\tan(\theta + 45^\circ)$

$$= \frac{\tan\theta + \tan 45^\circ}{1 - \tan\theta\tan 45^\circ}$$
$$= \frac{\tan\theta + 1}{1 - \tan\theta}$$
$$= \frac{\frac{1}{3} + 1}{1 - \frac{1}{3}}$$
$$= \frac{\left(\frac{4}{3}\right)}{\left(\frac{2}{3}\right)}$$
$$= \frac{4}{2}$$
$$= 2$$

11

Quadratic equations often arise naturally in trigonometric problems, largely because the relationships between functions usually rely on the squared expression.

a $2\sec^2 x + 7\tan x = 17$

Using $\sec^2 x = 1 + \tan^2 x$:

$$2\left(1 + \tan^2 x\right) + 7\tan x - 17 = 0$$
$$2\tan^2 x + 7\tan x - 15 = 0$$

Letting $u = \tan x$:

$$2u^2 + 7u - 15 = 0$$
$$(2u - 3)(u + 5) = 0$$
$$u = \frac{3}{2} \text{ or } u = -5$$

$\tan x = \frac{3}{2}$ or $\tan x = -5$

x is acute, so $\tan x > 0$

$\tan x = \frac{3}{2}$

b $\tan(225^\circ - x)$

$= \frac{\tan 225^\circ - \tan x}{1 + \tan 225^\circ \tan x}$

$= \frac{\tan 45^\circ - \tan x}{1 + \tan 45^\circ \tan x}$

(Using: $\tan 225^\circ = \tan(225^\circ - 180^\circ) = \tan 45^\circ$)

$= \frac{1 - \tan x}{1 + \tan x}$

$= \frac{1 - \frac{3}{2}}{1 + \frac{3}{2}}$

$= \frac{\left(-\frac{1}{2}\right)}{\left(\frac{5}{2}\right)}$

$= -\frac{1}{5}$

12 **a**

$$\begin{aligned} \sin(x + 30^\circ) &= 5\cos(x - 60^\circ) \\ \sin x \cos 30^\circ + \cos x \sin 30^\circ &= 5\cos x \cos 60^\circ + 5\sin x \sin 60^\circ \\ \sin x\,(\cos 30^\circ - 5\sin 60^\circ) &= \cos x\,(5\cos 60^\circ - \sin 30^\circ) \\ \frac{\cos x}{\sin x} &= \frac{\cos 30^\circ - 5\sin 60^\circ}{5\cos 60^\circ - \sin 30^\circ} \\ \cot x &= \frac{\frac{\sqrt{3}}{2} - 5\frac{\sqrt{3}}{2}}{\frac{5}{2} - \frac{1}{2}} \\ \cot x &= \frac{-4\sqrt{3}}{4} = -\sqrt{3} \end{aligned}$$

b Using the result from part **a**:

$\cot x = -\sqrt{3}$

$\tan x = -\frac{1}{\sqrt{3}}$

$x = -30^\circ$ or $x = -30^\circ + 180^\circ = 150^\circ$

13 **a**

$$\begin{aligned} \cos(x + 30^\circ) &= 2\sin x \\ \cos x \cos 30^\circ - \sin x \sin 30^\circ &= 2\sin x \\ \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x &= 2\sin x \\ \frac{\sqrt{3}}{2}\cos x &= \frac{5}{2}\sin x \\ \frac{\sin x}{\cos x} &= \frac{\sqrt{3}}{5} \\ \tan x &= \frac{\sqrt{3}}{5} \end{aligned}$$

$x = 19.1^\circ$ or $x = 19.1^\circ + 180^\circ = 199.1^\circ$

b $x = -30^\circ$ (out of required range) or $x = -30^\circ + 180^\circ = 150^\circ$ or $x = 150^\circ + 180^\circ = 330^\circ$.

Trigonometric ratios – particularly those using tangents – often force you to use double or triple-decker fractions, as in worked solution **14 c**. Keep calm and work carefully.

14 **b**

$$\begin{aligned} 2\tan(45^\circ - x) &= 3\tan x \\ 2\left(\frac{\tan 45^\circ - \tan x}{1+\tan 45^\circ \tan x}\right) &= 3\tan x \\ 2\left(\frac{1-\tan x}{1+\tan x}\right) &= 3\tan x \\ 2 - 2\tan x &= 3\tan x + 3\tan^2 x \end{aligned}$$

$3\tan^2 x + 5\tan x - 2 = 0$

Letting $u = \tan x$:

$3u^2 + 5u - 2 = 0$

$(3u-1)(u+2) = 0$

$u = \frac{1}{3}$ or $u = -2$

$\tan x = \frac{1}{3}$ or $\tan x = -2$

$x = 18.4^\circ$ or $x = -63.4^\circ + 180^\circ = 116.6^\circ$

c

$$\begin{aligned} \sin(x+60^\circ) &= 2\cos(x+45^\circ) \\ \sin x\cos 60^\circ + \cos x \sin 60^\circ &= 2\cos x\cos 45^\circ - 2\sin x \sin 45^\circ \\ \sin x(\cos 60^\circ + 2\sin 45^\circ) &= \cos x(2\cos 45^\circ - \sin 60^\circ) \\ \frac{\sin x}{\cos x} &= \frac{2\cos 45^\circ - \sin 60^\circ}{\cos 60^\circ + 2\sin 45^\circ} \\ \tan x &= \frac{2\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}}{\frac{1}{2} + 2\frac{\sqrt{2}}{2}} \\ \tan x &= \frac{2\sqrt{2} - \sqrt{3}}{1 + 2\sqrt{2}} \\ x &= 16.0^\circ \end{aligned}$$

15

$$\begin{aligned} \tan(x-45^\circ) + \cot x &= 2 \\ \frac{\tan x - \tan 45^\circ}{1 + \tan x \tan 45^\circ} + \frac{1}{\tan x} &= 2 \\ \frac{\tan x - 1}{\tan x + 1} + \frac{1}{\tan x} &= 2 \\ \frac{\tan x(\tan x - 1) + 1 + \tan x}{\tan x(\tan x + 1)} &= 2 \\ \tan^2 x - \tan x + 1 + \tan x &= 2\tan^2 x + 2\tan x \\ \tan^2 x + 2\tan x - 1 &= 0 \end{aligned}$$

Letting $u = \tan x$:

$u^2 + 2u - 1 = 0$

$$u = \frac{-2 \pm \sqrt{2^2 - 4\times 1 \times (-1)}}{2\times 1} = \frac{-2\pm\sqrt{8}}{2} = -1 \pm \sqrt{2}$$

$\tan x = -1 + \sqrt{2}$ or $\tan x = -1 - \sqrt{2}$

$x = 22.5^\circ$ or $x = -67.5^\circ + 180^\circ = 112.5^\circ$

16 $\cos 6x = \cos(5x + x) = \cos 5x \cos x - \sin 5x \sin x$ .... [1]

$\cos 4x = \cos(5x - x) = \cos 5x \cos x + \sin 5x \sin x$ .... [2]

[1] + [2]:

$$\begin{aligned} \cos 6x + \cos 4x &= 2\cos 5x\cos x + \sin 5x \sin x - \sin 5x \sin x \\ &= 2\cos 5x \cos x \end{aligned}$$

17 $\cos(x-y) = \cos x\cos y + \sin x \sin y$

Note that $p^2 + q^2 = (\sin x + \sin y)^2 + (\cos x + \cos y)^2$

$$
\begin{aligned}
&= \sin^2 x + 2\sin x \sin y + \sin^2 y + \cos^2 x + 2\cos x \cos y + \cos^2 y \\
&= \sin^2 x + \cos^2 x + \sin^2 y + \cos^2 y + 2(\cos x \cos y + \sin x \sin y) \\
&= 1 + 1 + 2(\cos x \cos y + \sin x \sin y) \\
p^2 + q^2 &= 2 + 2\cos(x - y) \\
\cos(x - y) &= \frac{p^2 + q^2 - 2}{2}
\end{aligned}
$$

EXERCISE 3C

1 **a** Using $\sin 2A = 2\sin A\cos A$:

$2\sin 28^\circ \cos 28^\circ = \sin(2 \times 28^\circ) = \sin 56^\circ$

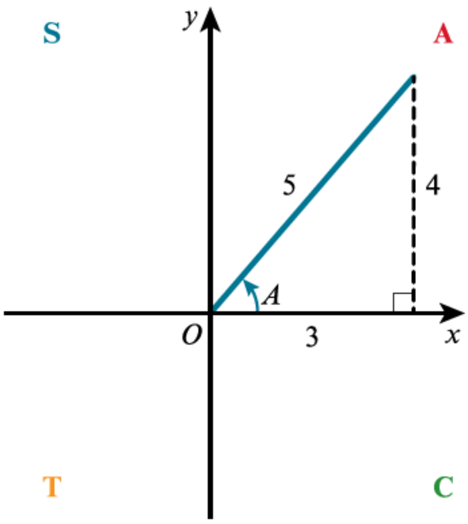
b Using $\cos 2A = 2\cos^2 A - 1$:

$2\cos^2 34^\circ - 1 = \cos(2 \times 34^\circ) = \cos 68^\circ$

c Using $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$:

$\dfrac{2\tan 17^\circ}{1 - \tan^2 17^\circ} = \tan(2 \times 17^\circ) = \tan 34^\circ$

2



$\tan x = \dfrac{4}{3}$ $\cos x = \dfrac{3}{5}$ $\sin x = \dfrac{4}{5}$

a $\sin 2x = 2\sin x\cos x$

$= 2\left(\dfrac{4}{5}\right)\left(\dfrac{3}{5}\right)$

$= \dfrac{24}{25}$

b $\tan 3x = \tan(2x + x)$

$= \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

But $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

$= \dfrac{2\left(\dfrac{4}{3}\right)}{1 - \left(\dfrac{4}{3}\right)^2}$

$= \dfrac{\left(\dfrac{8}{3}\right)}{1 - \dfrac{16}{9}}$

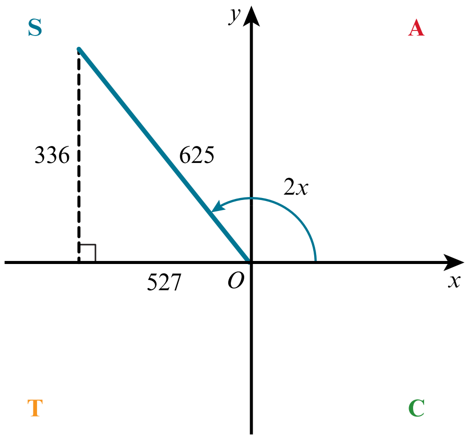
$= \dfrac{\left(\dfrac{8}{3}\right)}{\left(-\dfrac{7}{9}\right)}$

$= -\dfrac{8 \times 9}{7 \times 3}$

$= -\dfrac{24}{7}$

$$\text{So } \tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$
$$= \frac{-\frac{24}{7} + \frac{4}{3}}{1 - \left(-\frac{24}{7}\right)\left(\frac{4}{3}\right)}$$
$$= \frac{\left(-\frac{72}{21} + \frac{28}{21}\right)}{1 + \frac{96}{21}}$$
$$= \frac{28 - 72}{21 + 96}$$
$$= -\frac{44}{117}$$

3 $625^2 - 527^2 = 336^2$



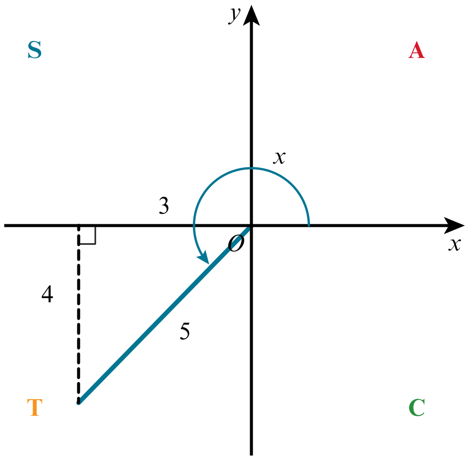
a From the diagram:

$$\sin 2x = \frac{336}{625}$$

b From the diagram:

$$\tan 2x = -\frac{336}{527}$$

4



a $\sin x = \frac{4}{5}$

$$\sin 2x = 2 \sin x \cos x$$
$$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)$$
$$= -\frac{24}{25}$$

c $\tan 2x = \dfrac{\sin 2x}{\cos 2x}$

$= \dfrac{2\sin x \cos x}{2\cos^2 x - 1}$

$= \dfrac{2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)}{2\left(-\frac{3}{5}\right)^2 - 1}$

$= \dfrac{\left(-\frac{24}{25}\right)}{\left(\frac{18}{25}\right) - 1}$

$= \dfrac{-24}{18 - 25}$

$= \dfrac{24}{7}$

5 $3\cos 2x + 17\sin x = 8$

$3\left(1 - 2\sin^2 x\right) + 17\sin x = 8$

$3 - 6\sin^2 x + 17\sin x = 8$

$6\sin^2 x - 17\sin x + 5 = 0$

Letting $u = \sin x$:

$6u^2 - 17u + 5 = 0$

$(3u - 1)(2u - 5) = 0$

$u = \dfrac{1}{3}$ or $u = \dfrac{5}{2}$

$\sin x = \dfrac{1}{3}$ or $\sin x = \dfrac{5}{2}\left(\text{not possible as} \dfrac{5}{2} > 1\right)$

6 **a** $2\sin 2\theta = \cos\theta$

$2(2\sin\theta\cos\theta) = \cos\theta$

$4\sin\theta\cos\theta = \cos\theta$

$\cos\theta(4\sin\theta - 1) = 0$

$\cos\theta = 0$

giving $\theta = 90^\circ$ or $\theta = 270^\circ$

or $\sin\theta = \dfrac{1}{4}$

giving $\theta = 14.5^\circ$ or $\theta = 180^\circ - 14.5^\circ = 165.5^\circ$

b $2\cos 2\theta + 3 = 4\cos\theta$

$2\left(2\cos^2\theta - 1\right) + 3 = 4\cos\theta$

$4\cos^2\theta - 2 + 3 = 4\cos\theta$

$4\cos^2\theta - 4\cos\theta + 1 = 0$

Letting $u = \cos\theta$:

$4u^2 - 4u + 1 = 0$

$(2u - 1)^2 = 0$

$u = \dfrac{1}{2}$

$\cos\theta = \dfrac{1}{2}$

$\theta = 60^\circ$ or $\theta = 360^\circ - 60^\circ = 300^\circ$

c $2\cos 2\theta + 1 = \sin\theta$

$2\left(1 - 2\sin^2\theta\right) + 1 = \sin\theta$

$2 - 4\sin^2\theta + 1 = \sin\theta$

$4\sin^2\theta + \sin\theta - 3 = 0$

Letting $u = \sin\theta$:

$$4u^2 + u - 3 = 0$$
$$(4u - 3)(u + 1) = 0$$
$$u = \frac{3}{4} \text{ or } u = -1$$
$$\sin\theta = \frac{3}{4}$$
giving $\theta = 48.6^\circ$ or $\theta = 180^\circ - 48.6^\circ = 131.4^\circ$

or $\sin\theta = -1$

giving $\theta = 270^\circ$

7

When there are multiple calculations to make in a question, you might find that you don't want to think about how to factorise. You can always use the quadratic formula instead. If you are not asked to factorise, you don't have to. However, if you can factorise, it is always a faster method. You are also less likely to make mistakes.

b
$$3\cos 2\theta + \cos\theta = 2$$
$$3\left(2\cos^2\theta - 1\right) + \cos\theta - 2 = 0$$
$$6\cos^2\theta - 3 + \cos\theta - 2 = 0$$
$$6\cos^2\theta + \cos\theta - 5 = 0$$
Letting $u = \cos\theta$:
$$6u^2 + u - 5 = 0$$
$$(6u - 5)(u + 1) = 0$$
$$u = \frac{5}{6} \text{ or } u = -1$$
$$\cos\theta = \frac{5}{6} \text{ or } \cos\theta = -1$$
$$\theta = 33.6^\circ \text{ or } \theta = 180^\circ$$

e
$$\tan 2\theta = 4\cot\theta$$
$$\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{4}{\tan\theta}$$
$$2\tan^2\theta = 4(1 - \tan^2\theta)$$
$$2\tan^2\theta = 4 - 4\tan^2\theta$$
$$6\tan^2\theta = 4$$
$$\tan^2\theta = \frac{2}{3}$$
$$\tan\theta = \sqrt{\frac{2}{3}} \text{ or } \tan\theta = -\sqrt{\frac{2}{3}}$$
$$\theta = 39.2^\circ \text{ or } \theta = -39.2^\circ + 180^\circ = 140.8^\circ$$
Note that $\theta = 90^\circ$ is also a solution.

Whenever you are working on a question that involves $\cos 2x$, you need to remember that there are three possible identities to use. Two of these identities contain only one of $\sin x$ or $\cos x$. This is usually the key point to help you decide which to use. If you need to convert to cosines, use the identity with cosines in it. Follow a similar process for sines, too.

8
$$\cos 2A = 2\cos^2 A - 1$$
$$\cos 4x = 2\cos^2 2x - 1$$
$$2\cos^2 2x = 1 + \cos 4x$$
$$\cos^2 2x = \frac{1 + \cos 4x}{2}$$

9 **a** $\cos 3x$
$= \cos(2x + x)$
$= \cos 2x \cos x - \sin 2x \sin x$
$= \left(2\cos^2 x - 1\right)\cos x - (2\sin x \cos x)\sin x$
$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x$
$= 2\cos^3 x - \cos x - 2\left(1 - \cos^2 x\right)\cos x$
$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$
$= 4\cos^3 x - 3\cos x$

b $\sin 3x$
$= \sin(2x + x)$
$= \sin 2x \cos x + \cos 2x \sin x$
$= (2\sin x \cos x)\cos x + \left(1 - 2\sin^2 x\right)\sin x$
$= 2\sin x \cos^2 x + \sin x - 2\sin^3 x$
$= 2\sin x\left(1 - \sin^2 x\right) + \sin x - 2\sin^3 x$
$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$
$= 3\sin x - 4\sin^3 x$

10

$$\begin{aligned}
\tan 2\theta + 2\tan\theta &= 3\cot\theta \\
\frac{2\tan\theta}{1 - \tan^2\theta} + 2\tan\theta &= \frac{3}{\tan\theta} \\
\frac{2\tan\theta + 2\tan\theta - 2\tan^3\theta}{1 - \tan^2\theta} &= \frac{3}{\tan\theta} \\
\tan\theta(4\tan\theta - 2\tan^3\theta) &= 3 - 3\tan^2\theta \\
4\tan^2\theta - 2\tan^4\theta &= 3 - 3\tan^2\theta \\
2\tan^4\theta - 7\tan^2\theta + 3 &= 0
\end{aligned}$$

Letting $u = \tan^2\theta$:

$$\begin{aligned}
2u^2 - 7u + 3 &= 0 \\
(2u - 1)(u - 3) &= 0
\end{aligned}$$

$u = \frac{1}{2}$ or $u = 3$

$\tan^2\theta = \frac{1}{2}$ or $\tan^2\theta = 3$

$\tan\theta = \pm\frac{1}{\sqrt{2}}$ or $\tan\theta = \pm\sqrt{3}$

$\theta = \pm 35.3^\circ$ or $\theta = \pm 60^\circ$

Negative solutions are out of the required range, so adding 180° to both negative results:

$\theta = 35.3^\circ$

or $\theta = -35.3^\circ + 180^\circ = 144.7^\circ$
or $\theta = 60^\circ$
or $\theta = -60^\circ + 180^\circ = 120^\circ$

11 **a** $\tan\theta + \cot\theta$

$$\begin{aligned}
&\equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\
&\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \\
&\equiv \frac{1}{\sin\theta\cos\theta} \\
&\equiv \frac{2}{2\sin\theta\cos\theta} \\
&\equiv \frac{2}{\sin 2\theta}
\end{aligned}$$

b Using the result from part **a**:

$\tan\frac{\pi}{12} + \cot\frac{\pi}{12}$
$= \frac{2}{\sin 2\left(\frac{\pi}{12}\right)}$
$= \frac{2}{\sin\left(\frac{\pi}{6}\right)}$
$= \frac{2}{\left(\frac{1}{2}\right)}$
$= 4$

12 **a** $2\operatorname{cosec} 2\theta \tan\theta$
$\equiv \frac{2\tan\theta}{\sin 2\theta}$
$\equiv \frac{2\tan\theta}{2\sin\theta\cos\theta}$
$\equiv \frac{\left(\frac{\sin\theta}{\cos\theta}\right)}{\sin\theta\cos\theta}$
$\equiv \frac{\sin\theta}{\sin\theta\cos^2\theta}$
$\equiv \frac{1}{\cos^2\theta}$
$\equiv \sec^2\theta$

b Using the result from part **a**:

$\frac{1}{2}\sec^2\theta = 2$
$\sec^2\theta = 4$
$\sec\theta = \pm 2$
$\cos\theta = \frac{1}{2}$
giving $\theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$
or $\cos\theta = -\frac{1}{2}$
giving $\theta = \frac{2\pi}{3}$ or $\theta = -\frac{2\pi}{3}$

13 **a** $\cos 4x + 4\cos 2x$
$= \cos(2 \times 2x) + 4\cos 2x$
$= 2\cos^2(2x) - 1 + 4\cos 2x$
$= 2\left(2\cos^2 x - 1\right)^2 - 1 + 4\left(2\cos^2 x - 1\right)$
$= 2(4\cos^4 x - 4\cos^2 x + 1) - 1 + 8\cos^2 x - 4$
$= 8\cos^4 x - 8\cos^2 x + 2 - 1 + 8\cos^2 x - 4$
$= 8\cos^4 x - 3$

b Using the result from part **a**:

$2\cos 4x + 8\cos 2x$
$= 2(\cos 4x + 4\cos 2x)$
$= 2\left(8\cos^4 x - 3\right) = 3$

$$8\cos^4 x - 3 = \frac{3}{2}$$
$$8\cos^4 x = \frac{9}{2}$$
$$\cos^4 x = \frac{9}{16}$$
$$\cos^2 x = \pm\sqrt{\frac{9}{16}} = \pm\frac{3}{4}$$
$$\cos^2 x > 0$$
$$\cos^2 x = \frac{3}{4}$$
$$\cos x = \pm\frac{\sqrt{3}}{2}$$
$$x = \pm\frac{\pi}{6} \text{ or } x = \pm\frac{5\pi}{6}$$

> Always remember that squared functions will be at least zero in value. This fact often helps you to exclude some impossible solutions.

14 **a** Letting $\theta = 18^\circ$:

$$\begin{aligned}\sin 3\theta &= \sin 54^\circ \\ &= \sin(90^\circ - 36^\circ) \\ &= \cos 36^\circ \\ &= \cos 2\theta\end{aligned}$$

So $\theta = 18^\circ$ is a solution of $\sin 3\theta = \cos 2\theta$.

b $\sin 3\theta$

$\equiv \sin(2\theta + \theta)$

$\equiv \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$\equiv 2\sin\theta\cos\theta\cos\theta + \left(1 - 2\sin^2\theta\right)\sin\theta$

$\equiv 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$

$\equiv 2\sin\theta\left(1 - \sin^2\theta\right) + \sin\theta - 2\sin^3\theta$

$\equiv 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$\equiv 3\sin\theta - 4\sin^3\theta$

$\cos 2\theta$

$\equiv 1 - 2\sin^2\theta$

c $\sin 3\theta = \cos 2\theta$

is equivalent to

$$3\sin\theta - 4\sin^3\theta = 1 - 2\sin^2\theta$$

$4\sin^3\theta - 2\sin^2\theta - 3\sin\theta + 1 = 0$

Letting $x = \sin\theta$:

$4x^3 - 2x^2 - 3x + 1 = 0$

$\theta = 18^\circ$ is a solution of $4\sin^3\theta - 2\sin^2\theta - 3\sin\theta + 1 = 0$

So $\sin 18^\circ$ is a solution of $4x^3 - 2x^2 - 3x + 1 = 0$.

d Let $\mathrm{f}(x) = 4x^3 - 2x^2 - 3x + 1$.

$\mathrm{f}(1) = 4(1)^3 - 2(1)^2 - 3(1) + 1 = 0$

So $x - 1$ is a factor of $\mathrm{f}(x)$ by the factor theorem.

Dividing $\mathrm{f}(x)$ by $x - 1$ using long division:

$$\begin{array}{r} 4x^2+2x-1 \\ x-1 \,\big)\, \overline{4x^3-2x^2-3x+1} \\ \underline{4x^3-4x^2} \qquad\qquad \\ 2x^2-3x \quad\;\; \\ \underline{2x^2-2x} \quad\;\; \\ -x+1 \\ \underline{-x+1} \\ 0 \end{array}$$

$(x-1)\left(4x^2+2x-1\right)=0$

$x=1 \text{ or } x=\dfrac{-2\pm\sqrt{2^2-4\times 4\times(-1)}}{2\times 4}=\dfrac{-2\pm\sqrt{20}}{8}=\dfrac{-1\pm\sqrt{5}}{4}$

$\sin 18^\circ$ is one of these solutions and $\sin 18^\circ > 0$.

$\sin 18^\circ \neq 1$

$\sin 18^\circ = \dfrac{-1+\sqrt{5}}{4}$

15

$$\begin{aligned} \cos 2\theta &> \cos\theta \\ 2\cos^2\theta - 1 &> \cos\theta \\ 2\cos^2\theta - \cos\theta - 1 &> 0 \\ (2\cos\theta+1)(\cos\theta-1) &> 0 \end{aligned}$$

$\cos\theta < -\dfrac{1}{2}$ or $\cos\theta > 1$.

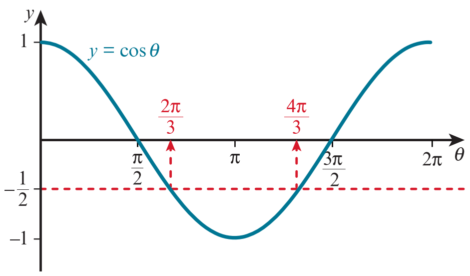
The second inequality has no solutions.

$\cos\theta < -\dfrac{1}{2}$

Critical values are:

$\cos\theta = -\dfrac{1}{2}$

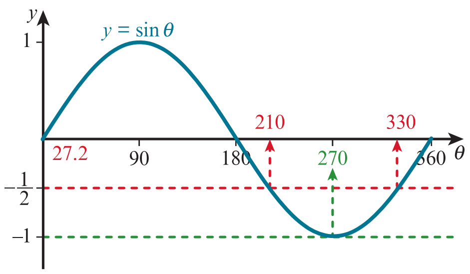
$\theta = \dfrac{2\pi}{3}$ or $\theta = 2\pi - \dfrac{2\pi}{3} = \dfrac{4\pi}{3}$



$\dfrac{2\pi}{3} < \theta < \dfrac{4\pi}{3}$

16

$$\begin{aligned} \cos 2\theta - 3\sin\theta - 2 &\geqslant 0 \\ 1 - 2\sin^2\theta - 3\sin\theta - 2 &\geqslant 0 \\ 2\sin^2\theta + 3\sin\theta + 1 &\leqslant 0 \\ (2\sin\theta+1)(\sin\theta+1) &\leqslant 0 \\ -1 \leqslant \sin\theta &\leqslant -\frac{1}{2} \end{aligned}$$



Critical values are:

$\sin\theta = -1$

$\theta = 270^\circ$

$\sin\theta = -\dfrac{1}{2}$

$\theta = 210^\circ$ or $\theta = 330^\circ$

$210^\circ \leqslant \theta \leqslant 330^\circ$

17 $A = 180^\circ - x - 3x = 180^\circ - 4x$

Using the sine rule:

$$\frac{\sin x}{b} = \frac{\sin(180^\circ - 4x)}{a}$$
$$a\sin x = b\sin(180^\circ - 4x)$$
$$a\sin x = b(\sin 180^\circ \cos 4x - \cos 180^\circ \sin 4x)$$
$$a\sin x = 0 - (-1)b\sin 4x$$
$$a\sin x = b\sin 4x$$
$$a\sin x = b(2\sin 2x\cos 2x)$$
$$a\sin x = 2b(2\sin x\cos x)\cos 2x$$
$$a\sin x = 4b\sin x\cos x\cos 2x$$

Dividing by $\sin x$:

$a = 4b\cos 2x\cos x$

18 **a** $\cos 3x = \cos(2x + x) = \cos 2x\cos x - \sin 2x\sin x$ ...... [1]

$\cos x = \cos(2x - x) = \cos 2x\cos x + \sin 2x\sin x$ ...... [2]

[1] + [2]:

$\cos 3x + \cos x = 2\cos 2x\cos x$

b Using $\cos 3x + \cos x = 2\cos 2x\cos x$:

$$\cos 3x + \cos 2x + \cos x > 0$$
$$(\cos 3x + \cos x) + \cos 2x > 0$$
$$2\cos 2x\cos x + \cos 2x > 0$$
$$\cos 2x\,(2\cos x + 1) > 0$$

$\cos 2x > 0$ and $\cos x > -\dfrac{1}{2}$ ...... [1]

or

$\cos 2x < 0$ and $\cos x < -\dfrac{1}{2}$ ...... [2]

$\cos x > -\dfrac{1}{2}$ when $x < 120^\circ$ or $x > 240^\circ$

$\cos x < -\dfrac{1}{2}$ when $120^\circ < x < 240^\circ$

$\cos 2x > 0$ when $0^\circ < 2x < 90^\circ$ or $270^\circ < 2x < 360^\circ$

i.e. when $0^\circ < x < 45^\circ$ or $135^\circ < x < 180^\circ$

$\cos 2x < 0$ when $45^\circ < x < 135^\circ$

[1] requires $x < 120^\circ$ and $0^\circ < x < 45^\circ$

so $0^\circ < x < 45^\circ$

[2] requires $120^\circ < x < 240^\circ$ and $45^\circ < x < 135^\circ$

so $120^\circ < x < 135^\circ$

19

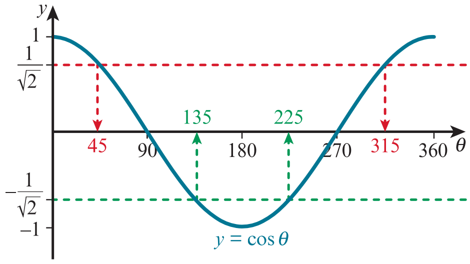
$$\cos 4\theta + 3\cos 2\theta + 1 < 0$$
$$\cos(2\times 2\theta) + 3\cos 2\theta + 1 < 0$$
$$2\cos^2 2\theta - 1 + 3(2\cos^2\theta - 1) + 1 < 0$$
$$2(2\cos^2\theta - 1)^2 - 1 + 6\cos^2\theta - 3 + 1 < 0$$
$$2(4\cos^4\theta - 4\cos^2\theta + 1) - 1 + 6\cos^2\theta - 2 < 0$$
$$8\cos^4\theta - 2\cos^2\theta - 1 < 0$$
$$(4\cos^2\theta + 1)(2\cos^2\theta - 1) < 0$$

The first bracket is always positive, so

$$\left(2\cos^2\theta - 1\right) < 0$$

$$\cos^2\theta < \frac{1}{2}$$

$$-\frac{1}{\sqrt{2}} < \cos\theta < \frac{1}{\sqrt{2}}$$



Critical values are:

$$\cos\theta = \frac{1}{\sqrt{2}} \qquad \cos\theta = -\frac{1}{\sqrt{2}}$$

$$\theta = 45^\circ,\ 315^\circ \qquad \theta = 135^\circ,\ 225^\circ$$

From the diagram:

$45^\circ < \theta < 135^\circ$ or $225^\circ < \theta < 315^\circ$

EXERCISE 3D

When proving identities, you will often find that one side includes more separate terms than the other. It is usually best to start with this side, because you can then work to simplify. If you start with the simpler side, you will often need to find 'tricks' to help you rewrite the expressions in a helpful way.

1 **b** $1-\tan^2 A$

$$\equiv 1-\frac{\sin^2 A}{\cos^2 A}$$

$$\equiv \frac{\cos^2 A}{\cos^2 A}-\frac{\sin^2 A}{\cos^2 A}$$

$$\equiv \frac{\cos^2 A-\sin^2 A}{\cos^2 A}$$

$$\equiv \frac{\cos 2A}{\cos^2 A}$$

$$\equiv \cos 2A \sec^2 A$$

f $\operatorname{cosec} 2A+\cot 2A$

$$\equiv \frac{1}{\sin 2A}+\frac{\cos 2A}{\sin 2A}$$

$$\equiv \frac{1+\cos 2A}{\sin 2A}$$

$$\equiv \frac{1+2\cos^2 A-1}{\sin 2A}$$

$$\equiv \frac{2\cos^2 A}{2\sin A\cos A}$$

$$\equiv \frac{\cos A}{\sin A}$$

$$\equiv \cot A$$

2 **c** $\dfrac{1-\tan^2 A}{1+\tan^2 A}$

$$\equiv \frac{1-\dfrac{\sin^2 A}{\cos^2 A}}{1+\dfrac{\sin^2 A}{\cos^2 A}}$$

$$\equiv \frac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}$$

$$\equiv \frac{\cos 2A}{1}$$

$$\equiv \cos 2A$$

g $\dfrac{\cos 2A+9\cos A+5}{4+\cos A}$

$$\equiv \frac{2\cos^2 A-1+9\cos A+5}{4+\cos A}$$

$$\equiv \frac{2\cos^2 A+9\cos A+4}{4+\cos A}$$

$$\equiv \frac{(2\cos A+1)(\cos A+4)}{4+\cos A}$$

$$\equiv 2\cos A+1$$

3 **a** Using the fact that $4A=2\times 2A$:

$\frac{\sin 4A}{\sin A}$

$\equiv \frac{2\sin 2A\cos 2A}{\sin A}$

$\equiv \frac{2(2\sin A\cos A)(2\cos^2 A - 1)}{\sin A}$

$\equiv 4\cos A(2\cos^2 A - 1)$

$\equiv 8\cos^3 A - 4\cos A$

b Using the fact that $4A = 2 \times 2A$:

$\cos 4A$

$\equiv 2\cos^2(2A) - 1$

$\equiv 2(2\cos^2 A - 1)^2 - 1$

$\equiv 2(4\cos^4 A - 4\cos^2 A + 1) - 1$

$\equiv 8\cos^4 A - 8\cos^2 A + 2 - 1$

$\equiv 8\cos^4 A - 8\cos^2 A + 1$

Again, using the fact that $4A = 2 \times 2A$:

$4\cos 2A$

$\equiv 4\left(2\cos^2 A - 1\right)$

$\equiv 8\cos^2 A - 4$

$\cos 4A + 4\cos 2A \equiv 8\cos^4 A - 8\cos^2 A + 1 + 8\cos^2 A - 4$

$\equiv 8\cos^4 A - 3$

4 $8\sin^2 x\cos^2 x \equiv 2(2\sin x\cos x)^2 \equiv 2\sin^2 2x$

But $\cos 4x \equiv 1 - 2\sin^2 2x$

So $2\sin^2 2x \equiv 1 - \cos 4x$

Giving:

$8\sin^2 x\cos^2 x \equiv 1 - \cos 4x$

5 $(2\sin A + \cos A)^2$

$\equiv 4\sin^2 A + 4\sin A\cos A + \cos^2 A$

$\equiv 3\sin^2 A + 2(2\sin A\cos A) + \sin^2 A + \cos^2 A$

$\equiv 3\left(\frac{1}{2} - \frac{1}{2}\cos 2A\right) + 2\sin 2A + 1$

$\equiv 2\sin 2A - \frac{3}{2}\cos 2A + \frac{5}{2}$

$\equiv \frac{1}{2}(4\sin 2A - 3\cos 2A + 5)$

6 $\cos 2x \equiv \cos(3x - x) \equiv \cos 3x\cos x + \sin 3x\sin x$ .... [1]

$\cos 4x \equiv \cos(3x + x) \equiv \cos 3x\cos x - \sin 3x\sin x$ .... [2]

[1] − [2]:

$\cos 2x - \cos 4x \equiv \cos 3x\cos x + \sin 3x\sin x - (\cos 3x\cos x - \sin 3x\sin x)$

$\cos 2x - \cos 4x \equiv 2\sin 3x\sin x$

EXERCISE 3E

1 **a** $15\sin\theta - 8\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Equating coefficients of $\sin\theta$: $R\cos\alpha = 15$ .... [1]

Equating coefficients of $\cos\theta$: $R\sin\alpha = 8$ .... [2]

[2] ÷ [1]:

$\tan\alpha = \frac{8}{15} \Rightarrow \alpha = 28.07^\circ$

$R^2 = 15^2 + 8^2$

$R = \sqrt{289} = 17$

$15\sin\theta - 8\cos\theta = 17\sin(\theta - 28.07^\circ)$

b $15\sin\theta - 8\cos\theta = 10$

$17\sin(\theta - 28.07^\circ) = 10$

$\sin(\theta - 28.07^\circ) = \frac{10}{17}$

$\theta - 28.07^\circ = 36.0$ or $\theta - 28.07^\circ = 180^\circ - 36.0^\circ$

$= 144.0^\circ$

$\theta = 64.1^\circ$ $\theta = 172^\circ$

2 **a** $2\cos\theta - 3\sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Equating coefficients of $\cos\theta$: $R\cos\alpha = 2$ .... [1]

Equating coefficients of $\sin\theta$: $R\sin\alpha = 3$ .... [2]

[2] ÷[1]:

$\tan\alpha = \frac{3}{2} \Rightarrow \alpha = 56.31^\circ$

$R^2 = 2^2 + 3^2 = 13$

$R = \sqrt{13}$

$2\cos\theta - 3\sin\theta = \sqrt{13}\cos(\theta + 56.31^\circ)$

b $\sqrt{13}\cos(\theta + 56.31^\circ) = 1.3$

$\cos(\theta + 56.31^\circ) = \frac{1.3}{\sqrt{13}}$

$\theta + 56.31^\circ = 68.8657....^\circ$

or $\theta + 56.31^\circ = 360^\circ - 68.8657....^\circ$

$= 291.134...^\circ$

$\theta = 234.8^\circ$ or $\theta = 12.6^\circ$

3 **a** $15\sin\theta - 8\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Equating coefficients of $\sin\theta$: $R\cos\alpha = 15$ .... [1]

Equating coefficients of $\cos\theta$: $R\sin\alpha = 8$ .... [2]

[2] ÷ [1]:

$\tan\alpha = \frac{8}{15} \Rightarrow \alpha = 28.07^\circ$

$R^2 = 15^2 + 8^2 = 289$

$R = \sqrt{289} = 17$

$15\sin\theta - 8\cos\theta = 17\sin(\theta - 28.07^\circ)$

b $17\sin(\theta - 28.07^\circ) = 3$

$\sin(\theta - 28.07^\circ) = \frac{3}{17}$

$\theta - 28.07^\circ = 10.1642...^\circ$ or

$\theta - 28.07^\circ = 169.8357...^\circ$

$\theta = 38.2^\circ$ or $\theta = 197.9^\circ$

c $15\sin\theta - 8\cos\theta = 17\sin(\theta - 28.07^\circ)$

$30\sin\theta - 16\cos\theta = 34\sin(\theta - 28.07^\circ)$

Greatest value when $\sin(\theta - 28.07^\circ) = 1$

Greatest value $= 34$

4 **a** $4\sin\theta - 6\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Equating coefficients of $\sin\theta$: $R\cos\alpha = 4$ .... [1]

Equating coefficients of $\cos\theta$: $R\sin\alpha = 6$ .... [2]

[2] ÷[1]:

$\tan\alpha = \frac{6}{4} \Rightarrow \alpha = 56.31^\circ$

$R^2 = 4^2 + 6^2 = 52$

$R = \sqrt{52} = 2\sqrt{13}$

$4\sin\theta - 6\cos\theta = 2\sqrt{13}\sin(\theta - 56.31^\circ)$

b $2\sqrt{13}\sin(\theta - 56.31^\circ) = 3$

$\sin(\theta - 56.31^\circ) = \frac{3}{2\sqrt{13}}$

$\theta - 56.31^\circ = 24.583 \ldots\ldots^\circ$ or $\theta - 56.31^\circ = 155.416 \ldots\ldots^\circ$

$\theta = 80.9^\circ$ or $\theta = 211.7^\circ$ (out of the required range)

c $(4\sin\theta - 6\cos\theta)^2 - 3$

$= \{2\sqrt{13}\sin(\theta - 56.31^\circ)\}^2 - 3$

$= 52\sin^2(\theta - 56.31^\circ) - 3$

$0 \leqslant \sin^2(\theta - 56.31^\circ) \leqslant 1$

$-3 \leqslant 52\sin^2(\theta - 56.31^\circ) - 3 \leqslant 49$

Maximum $= 49$

Minimum $= -3$

5 **a** $3\sin\theta + 4\cos\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

Equating coefficients of $\sin\theta$: $R\cos\alpha = 3$ .... [1]

Equating coefficients of $\cos\theta$: $R\sin\alpha = 4$ .... [2]

[2] ÷ [1]:

$\tan\alpha = \frac{4}{3} \Rightarrow \alpha = 53.13^\circ$

$R^2 = 3^2 + 4^2 = 25$

$R = 5$

$3\sin\theta + 4\cos\theta = 5\sin(\theta + 53.13^\circ)$

b $5\sin(\theta + 53.13^\circ) = 2$

$\sin(\theta + 53.13^\circ) = \frac{2}{5}$

$\theta + 53.13^\circ = 23.578\ldots^\circ$

or $\theta + 53.13^\circ = 180^\circ - 23.578\ldots^\circ$

$= 156.4218\ldots^\circ$

or $\theta + 53.13^\circ = 23.578^\circ\ldots + 360^\circ$

$= 383.578\ldots^\circ$

$\theta = -29.6^\circ$ (out of the required range)

$\theta = 103.3^\circ$

$\theta = 330.4^\circ$

Questions using this particular trigonometric method usually require you to add an angle at the very end, as has been the case in almost every question so far. When you add this angle, it will sometimes give you a solution out of the required range. If this happens, another solution will appear at the opposite extreme.

c $3\sin\theta + 4\cos\theta = 5\sin(\theta + 53.13^\circ)$

$3\sin\theta + 4\cos\theta + 3 = 5\sin(\theta + 53.13^\circ) + 3$

Minimum value $= -5 + 3 = -2$

6 **a** $\cos\theta + \sqrt{3}\sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Equating coefficients of $\cos\theta$: $R\cos\alpha = 1$ .... [1]

Equating coefficients of $\sin\theta$: $R\sin\alpha = \sqrt{3}$ .... [2]

[2] ÷ [1]:

$\tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$

$R^2 = 1^2 + \left(\sqrt{3}\right)^2 = 4$

$R = 2$

$\cos\theta + \sqrt{3}\sin\theta = 2\cos\left(\theta - \frac{\pi}{3}\right)$

b

$$\cos\theta + \sqrt{3}\sin\theta = 2\cos\left(\theta - \frac{\pi}{3}\right)$$

$$\frac{1}{\left(\cos\theta + \sqrt{3}\sin\theta\right)^2} = \frac{1}{2^2\cos^2\left(\theta - \frac{\pi}{3}\right)}$$

$$= \frac{1}{4}\sec^2\left(\theta - \frac{\pi}{3}\right)$$

7 **a** $8\sin 2\theta + 4\cos 2\theta = R\sin 2\theta\cos\alpha + R\cos 2\theta\sin\alpha$

Equating coefficients of $\sin 2\theta$: $R\cos\alpha = 8$ .... [1]

Equating coefficients of $\cos 2\theta$: $R\sin\alpha = 4$ .... [2]

[2] ÷ [1]:

$\tan\alpha = \frac{1}{2} \Rightarrow \alpha = 26.57^\circ$

$R^2 = 8^2 + 4^2 = 80$

$R = \sqrt{80} = 4\sqrt{5}$

$8\sin 2\theta + 4\cos 2\theta = 4\sqrt{5}\sin(2\theta + 26.57^\circ)$

b $4\sqrt{5}\sin(2\theta + 26.57^\circ) = 3$

$\sin(2\theta + 26.57^\circ) = \frac{3}{4\sqrt{5}}$

$2\theta + 26.57^\circ = 19.597\ldots^\circ \Rightarrow \theta = -3.5^\circ$ (out of the required range)

or $2\theta + 26.57^\circ = 180^\circ - 19.597\ldots^\circ = 160.40251\ldots^\circ \Rightarrow \theta = 66.9^\circ$

or $2\theta + 26.57^\circ = 19.597\ldots^\circ + 360^\circ = 379.597\ldots^\circ \Rightarrow \theta = 176.5^\circ$

or $2\theta + 26.57^\circ = 160.40251\ldots^\circ + 360^\circ = 520.40251\ldots^\circ \Rightarrow \theta = 246.9^\circ$

or $2\theta + 26.57^\circ = 379.597\ldots^\circ + 360^\circ = 739.597\ldots^\circ \Rightarrow \theta = 356.5^\circ$

c

$$8\sin 2\theta + 4\cos 2\theta = 4\sqrt{5}\sin(2\theta + 26.57^\circ)$$

$$\frac{10}{(8\sin 2\theta + 4\cos 2\theta)^2} = \frac{10}{\left(4\sqrt{5}\sin(2\theta + 26.57^\circ)\right)^2}$$

$$= \frac{10}{80\sin^2(2\theta + 26.57^\circ)}$$

Least value when the denominator is largest.

Least value $= \frac{10}{80} = \frac{1}{8}$

> Although the sine and cosine functions are restricted to outputs between 1 and −1, you will find that the range is even narrower if there are any squared functions. Always watch out for this.

8 **a** $\cos\theta - \sqrt{2}\sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Equating coefficients of $\cos\theta$: $R\cos\alpha = 1$ .... [1]

Equating coefficients of $\sin\theta$: $R\sin\alpha = \sqrt{2}$ .... [2]

[2] $\div$ [1]:

$\tan\alpha = \sqrt{2} \Rightarrow \alpha = 54.74^\circ$

$R^2 = 1^2 + \left(\sqrt{2}\right)^2 = 3$

$R = \sqrt{3}$

$\cos\theta - \sqrt{2}\sin\theta = \sqrt{3}\cos(\theta + 54.74^\circ)$

b $\sqrt{3}\cos(\theta + 54.74^\circ) = -1$

$$\cos(\theta + 54.74^\circ) = -\frac{1}{\sqrt{3}}$$

$$\theta + 54.74^\circ = 125.264\ldots^\circ \Rightarrow \theta = 70.5^\circ$$

or $\theta + 54.74^\circ = 360^\circ - 125.264\ldots^\circ$

$= 234.735\ldots^\circ = 180^\circ$

c $$\frac{1}{\left(\sqrt{2}\cos\theta - 2\sin\theta\right)^2}$$

$$= \frac{1}{\left(\sqrt{2}\right)^2\left(\cos\theta - \sqrt{2}\sin\theta\right)^2}$$

$$= \frac{1}{2\left(\sqrt{3}\cos(\theta + 54.74^\circ)\right)^2}$$

$$= \frac{1}{6\cos^2(\theta + 54.74^\circ)}$$

Lease value when the denominator is largest.

Least value $= \dfrac{1}{6}$

9 **a** $\cos\theta - \sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Equating coefficients of $\cos\theta$: $R\cos\alpha = 1$ .... [1]

Equating coefficients of $\sin\theta$: $R\sin\alpha = 1$ .... [2]

[2] $\div$ [1]:

$\tan\alpha = 1 \Rightarrow \alpha = \dfrac{\pi}{4}$

$R^2 = 1^2 + 1^2 = 2$

$R = \sqrt{2}$

$\cos\theta - \sin\theta = \sqrt{2}\cos\left(\theta + \dfrac{\pi}{4}\right)$

b Using the result from part **a**, the given equation is equivalent to:

$$\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right) = \frac{1}{2}\sqrt{6}$$

$$\cos\left(\theta + \frac{\pi}{4}\right) = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$$

$\theta + \dfrac{\pi}{4} = \dfrac{\pi}{6} \Rightarrow \theta = \dfrac{\pi}{6} - \dfrac{\pi}{4} = -\dfrac{\pi}{12}$ (out of the required range)

or $\theta + \dfrac{\pi}{4} = 2\pi - \dfrac{\pi}{6} = \dfrac{11\pi}{6} \Rightarrow \theta = \dfrac{11\pi}{6} - \dfrac{\pi}{4} = \dfrac{19\pi}{12}$

or $\theta + \dfrac{\pi}{4} = \dfrac{\pi}{6} + 2\pi = \dfrac{13\pi}{6} \Rightarrow \theta = \dfrac{13\pi}{6} - \dfrac{\pi}{4} = \dfrac{23\pi}{12}$

c $\sqrt{2}\cos\left(\theta + \dfrac{\pi}{4}\right) = k$ will only have solutions if k lies between the maximum and minimum values of $\sqrt{2}\cos\left(\theta + \dfrac{\pi}{4}\right)$,

i.e. $-\sqrt{2} \leqslant k \leqslant \sqrt{2}$.

10 **a** $\sin\theta - 3\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Equating coefficients of $\sin\theta$: $R\cos\alpha = 1$ .... [1]

Equating coefficients of $\cos\theta$: $R\sin\alpha = 3$ .... [2]

[2] $\div$ [1]:

$\tan\alpha = 3 \Rightarrow \alpha = 71.57^\circ$
$R^2 = 1^2 + 3^2 = 10$
$R = \sqrt{10}$
$\sin\theta - 3\cos\theta = \sqrt{10}\sin(\theta - 71.57^\circ)$

b $\sqrt{10}\sin(\theta - 71.57^\circ) = -2$

$$\sin(\theta - 71.57^\circ) = -\frac{2}{\sqrt{10}}$$

$$\theta - 71.57^\circ = -39.231\ldots^\circ \Rightarrow \theta = 32.3^\circ$$

or $\quad \theta - 71.57^\circ = 180^\circ - (-39.231\ldots^\circ) = 219.231\ldots^\circ \Rightarrow \theta = 290.8^\circ$

c $\quad \sin 2\theta - 3\cos 2\theta = \sqrt{10}\sin(2\theta - 71.57^\circ)$
$1 + \sin 2\theta - 3\cos 2\theta = 1 + \sqrt{10}\sin(2\theta - 71.57^\circ)$
Greatest possible value occurs when

$\sin(2\theta - 71.57^\circ) = 1$
$2\theta - 71.57^\circ = 90^\circ$
$\theta = 80.8^\circ$

Greatest possible value is $1 + \sqrt{10}$

11 **a** $\sqrt{5}\cos\theta + 2\sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Equating coefficients of $\cos\theta$: $R\cos\alpha = \sqrt{5}$ .... [1]

Equating coefficients of $\sin\theta$: $R\sin\alpha = 2$ .... [2]

[2] $\div$ [1]:

$\tan\alpha = \frac{2}{\sqrt{5}} \Rightarrow \alpha = 41.81^\circ$
$R^2 = \left(\sqrt{5}\right)^2 + 2^2 = 9$
$R = 3$
$\sqrt{5}\cos\theta + 2\sin\theta = 3\cos(\theta - 41.81^\circ)$

b $3\cos(\theta - 41.81^\circ) = 3$
$\cos(\theta - 41.81^\circ) = 1$
$\theta - 41.81^\circ = 0$
$\theta = 41.8^\circ$

c $3\cos\left(\frac{1}{2}\theta - 41.81^\circ\right) = -1$

$$\cos\left(\frac{1}{2}\theta - 41.81^\circ\right) = -\frac{1}{3}$$

$$\frac{1}{2}\theta - 41.81^\circ = 109.471\ldots^\circ \Rightarrow \theta = 302.6^\circ$$

12 **a** $3\sec\theta + 4\operatorname{cosec}\theta = 2\operatorname{cosec} 2\theta$

$$\frac{3}{\cos\theta} + \frac{4}{\sin\theta} = \frac{2}{\sin 2\theta}$$

$$\frac{3\sin\theta + 4\cos\theta}{\sin\theta\cos\theta} = \frac{2}{2\sin\theta\cos\theta}$$

$$3\sin\theta + 4\cos\theta = 1$$

b $3\sin\theta + 4\cos\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

Equating coefficients of $\sin\theta$: $R\cos\alpha = 3$ .... [1]

Equating coefficients of $\cos\theta$: $R\sin\alpha = 4$ .... [2]

[2] $\div$ [1]:

$\tan\alpha = \frac{4}{3} \Rightarrow \alpha = 53.13^\circ$
$R^2 = 3^2 + 4^2 = 25$
$R = 5$
$3\sin\theta + 4\cos\theta = 5\sin(\theta + 53.13^\circ)$

c $3\sin\theta + 4\cos\theta = 1$

$5\sin(\theta + 53.13^\circ) = 1$

$\sin(\theta + 53.13^\circ) = \frac{1}{5}$

$\theta + 53.13^\circ = 11.5369\ldots^\circ \Rightarrow \theta = -41.6^\circ$ (out of the required range)

or $\theta + 53.13^\circ = 180^\circ - 11.5369\ldots^\circ = 168.463\ldots^\circ \Rightarrow \theta = 115.3^\circ$

or $\theta + 53.13^\circ = 11.5369\ldots^\circ + 360^\circ = 371.5369\ldots^\circ \Rightarrow \theta = 318.4^\circ$

13 **a** $\sin(\theta + 30^\circ) + \cos\theta$

$= \sin\theta\cos 30^\circ + \cos\theta\sin 30^\circ + \cos\theta$

$= \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta + \cos\theta$

$= \frac{\sqrt{3}}{2}\sin\theta + \frac{3}{2}\cos\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

Equating coefficients of $\sin\theta$: $R\cos\alpha = \frac{\sqrt{3}}{2}$ ..... [1]

Equating coefficients of $\cos\theta$: $R\sin\alpha = \frac{3}{2}$ .... [2]

[2] ÷ [1]:

$\tan\alpha = \sqrt{3} \Rightarrow \alpha = 60^\circ$

$R^2 = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = \frac{3}{4} + \frac{9}{4} = 3$

$R = \sqrt{3}$

$\sin(\theta + 30^\circ) + \cos\theta = \sqrt{3}\sin(\theta + 60^\circ)$

b $\sqrt{3}\sin(\theta + 60^\circ) = 1$

$\sin(\theta + 60^\circ) = \frac{1}{\sqrt{3}}$

$\theta + 60^\circ = 35.3^\circ \Rightarrow \theta = -24.7$ (out of the required range)

or $\theta + 60^\circ = 180^\circ - 35.3^\circ = 144.7^\circ \Rightarrow \theta = 84.7^\circ$

or $\theta + 60^\circ = 35.3^\circ + 360^\circ = 395.3^\circ \Rightarrow \theta = 335.3^\circ$

14 **a** $7\sin^2\theta + 9\cos^2\theta + 4\sin\theta\cos\theta + 2$

$= 7\left(\sin^2\theta + \cos^2\theta\right) + 2\cos^2\theta + 2(2\sin\theta\cos\theta) + 2$

$= 2\cos^2\theta + 2\sin 2\theta + 9$ (using $\sin^2\theta + \cos^2\theta \equiv 1$)

$= 2\cos^2\theta - 1 + 2\sin 2\theta + 10$

$= \cos 2\theta + 2\sin 2\theta + 10$ (using $2\cos^2\theta - 1 \equiv \cos 2\theta$)

$\cos 2\theta + 2\sin 2\theta = R\cos(2\theta - \alpha)$

$\cos 2\theta + 2\sin 2\theta = R\cos 2\theta\ \cos\alpha + R\ \sin 2\theta\sin\alpha$

$R\ \cos\alpha = 1$

$R\ \sin\alpha = 2$

$R^2 = 1^2 + 2^2 = 5$

$R = \sqrt{5}$

$\tan\alpha = 2 \Rightarrow \alpha = 63.4^\circ$

$\cos 2\theta + 2\sin 2\theta + 10 = \sqrt{5}\cos(2\theta - 63.4^\circ) + 10$

Maximum $= 10 + \sqrt{5}$

Minimum $= 10 - \sqrt{5}$

b $7\sin^2\theta + 9\cos^2\theta + 4\sin\theta\ \cos\theta = 10$

$7\sin^2\theta + 9\cos^2\theta + 4\sin\theta\ \cos\theta + 2 = 12$

From the working in part **a**:

$\sqrt{5}\cos(2\theta - 63.4^\circ) + 10 = 12$

$\cos(2\theta - 63.4^\circ) = \frac{2}{\sqrt{5}}$

$2\theta - 63.4^\circ = 26.56505^\circ$

or $2\theta - 63.4^\circ = 333.43499^\circ$

or $2\theta - 63.4 = 26.56505^\circ + 360^\circ = 386.56505^\circ$

or $2\theta - 63.4 = -26.56505^\circ$ (included because the addition of 63.4° brings it into the range)

or $\theta = 18.4^\circ, 45^\circ, 198.4^\circ, 225^\circ$

END-OF-CHAPTER REVIEW EXERCISE 3

1 You can get to the graph required by transforming the graph of $y = \sec x$:

$y = \sec x$

$\downarrow$ Translation 90° in the positive x-direction.

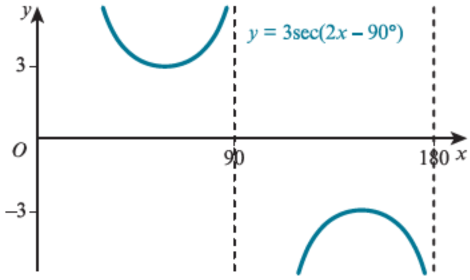
$y = \sec(x - 90^\circ)$

$\downarrow$ Stretch, factor $\frac{1}{2}$, parallel to the x-axis.

$y = \sec(2x - 90^\circ)$

$\downarrow$ Stretch, factor 3, parallel to the y-axis.

$y = 3\sec(2x - 90^\circ)$



If you are not comfortable with successive graph transformations, use a graph plotter to work through the various stages shown in worked solution **1**. It is helpful to watch the graph build, step-by-step. Simply enter each of the equations in turn, think about what has changed in the equation and match that with the changes you see on the screen.

2 $\operatorname{cosec}\theta = 3\sin\theta + \cot\theta$

$$\frac{1}{\sin\theta} = 3\sin\theta + \frac{\cos\theta}{\sin\theta}$$

Multiplying both sides by $\sin\theta$:

$$1 = 3\sin^2\theta + \cos\theta$$

$$3(1 - \cos^2\theta) + \cos\theta - 1 = 0$$

$$3 - 3\cos^2\theta + \cos\theta - 1 = 0$$

$$3\cos^2\theta - \cos\theta - 2 = 0$$

Letting $u = \cos\theta$:

$$3u^2 - u - 2 = 0$$

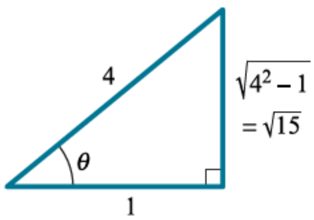
$$(3u + 2)(u - 1) = 0$$

$$u = -\frac{2}{3} \text{ or } u = 1$$

$$\cos\theta = -\frac{2}{3} \text{ or } \cos\theta = 1$$

$\theta = 131.8^\circ$ or $\theta = 0^\circ$ (just out of the required range)

3



$$\cos A = \frac{1}{4} \qquad \sin A = -\frac{\sqrt{15}}{4}$$

$$\sin 2A = 2\sin A\cos A$$

$$= 2\left(-\frac{\sqrt{15}}{4}\right)\left(\frac{1}{4}\right)$$

$$= -\frac{\sqrt{15}}{8}$$

4 $2\tan^2 x + \sec x = 1$

$2(\sec^2 x - 1) + \sec x = 1$

$2\sec^2 x + \sec x - 3 = 0$

Letting $u = \sec x$:

$2u^2 + u - 3 = 0$

$(2u + 3)(u - 1) = 0$

$u = -\frac{3}{2}$ or $u = 1$

$\sec x = -\frac{3}{2}$ $\sec x = 1$

$\cos x = -\frac{2}{3}$ $\cos x = 1$

$x = 131.8^\circ$ or $x = 360^\circ - 131.8^\circ = 228.2^\circ$

$x = 0^\circ$ or $x = 360^\circ$

5 $2\cot^2 x + 5\operatorname{cosec} x = 10$

$2\left(\operatorname{cosec}^2 x - 1\right) + 5\operatorname{cosec} x - 10 = 0$

$2\operatorname{cosec}^2 x + 5\operatorname{cosec} x - 12 = 0$

Letting $u = \operatorname{cosec} x$:

$2u^2 + 5u - 12 = 0$

$(2u - 3)(u + 4) = 0$

$u = \frac{3}{2}$ or $u = -4$

$\operatorname{cosec} x = \frac{3}{2}$ or $\operatorname{cosec} x = -4$

$\sin x = \frac{2}{3}$ or $\sin x = -\frac{1}{4}$

$x = 41.8^\circ$ or $x = -14.5^\circ$ (out of the required range)

$x = 180^\circ - 41.8^\circ = 138.2^\circ$ or $x = 180^\circ - (-14.5^\circ) = 194.5^\circ$ or $x = -14.5^\circ + 360 = 345.5^\circ$

6 **a** $\sin(x + 60^\circ) + \cos(x + 30^\circ)$

$\equiv \sin x \cos 60^\circ + \cos x \sin 60^\circ + \cos x \cos 30^\circ - \sin x \sin 30^\circ$

$\equiv \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x + \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$

$\equiv \sqrt{3}\cos x$

b $\sqrt{3}\cos x = \frac{3}{2}$

$\cos x = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$

$x = 30^\circ$ or $x = 360^\circ - 30^\circ = 330^\circ$

7 **a** $\sin(60^\circ - x) + \cos(30^\circ - x)$

$\equiv \sin 60^\circ \cos x - \cos 60^\circ \sin x + \cos 30^\circ \cos x + \sin 30^\circ \sin x$

$\equiv \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x$

$\equiv \sqrt{3}\cos x$

b $\frac{2}{5}\sec x = \sqrt{3}\cos x$

$\frac{2}{5\cos x} = \sqrt{3}\cos x$

$\cos^2 x = \frac{2}{5\sqrt{3}}$

$\cos x = \sqrt{\frac{2}{5\sqrt{3}}}$ or $\cos x = -\sqrt{\frac{2}{5\sqrt{3}}}$

$x = 61.3^\circ$ or $x = 360^\circ - 61.3^\circ = 298.7^\circ$

$x = 118.7^\circ$ or $x = 360^\circ - 118.7^\circ = 241.3^\circ$

8 **i**

$$\tan(x+45) = 6\tan x$$
$$\frac{\tan x + \tan 45}{1 - \tan x \tan 45} = 6\tan x$$
$$\frac{1+\tan x}{1-\tan x} = 6\tan x$$
$$1 + \tan x = 6\tan x - 6\tan^2 x$$
$$6\tan^2 - 5\tan x + 1 = 0$$

ii $6\tan^2 - 5\tan x + 1 = 0$

Letting $u = \tan x$:

$$6u^2 - 5u + 1 = 0$$
$$(2u-1)(3u-1) = 0$$

$u = \frac{1}{2}$ or $u = \frac{1}{3}$

$\tan x = \frac{1}{2}$ $\qquad$ $\tan x = \frac{1}{3}$

$x = 26.6^\circ$ $\qquad$ $x = 18.4^\circ$

9 **a** $\tan(x+45^\circ) - \tan(45^\circ - x)$

$$\equiv \frac{\tan x + \tan 45^\circ}{1 - \tan x \tan 45^\circ} - \frac{\tan 45^\circ - \tan x}{1 + \tan 45^\circ \tan x}$$
$$\equiv \frac{\tan x + 1}{1 - \tan x} - \frac{1 - \tan x}{1 + \tan x}$$
$$\equiv \frac{(\tan x + 1)^2 - (1 - \tan x)^2}{(1 - \tan x)(1 + \tan x)}$$
$$\equiv \frac{\tan^2 x + 2\tan x + 1 - (1 - 2\tan x + \tan^2 x)}{1 - \tan^2 x}$$
$$\equiv \frac{\tan^2 x + 2\tan x + 1 - 1 + 2\tan x - \tan^2 x}{1 - \tan^2 x}$$
$$\equiv \frac{4\tan x}{1 - \tan^2 x}$$
$$\equiv 2\frac{2\tan x}{1 - \tan^2 x}$$
$$\equiv 2\tan 2x$$

b $2\tan 2x = 6$

$\tan 2x = 3$

$2x = 71.565...^\circ$ or $2x = 71.565...^\circ + 180^\circ = 251.565...^\circ$

$x = 35.8^\circ$ $\qquad$ $x = 125.8^\circ$

10 **i** $3\cos\theta + \sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Equating coefficients of $\cos\theta$: $R\cos\alpha = 3$ .... [1]

Equating coefficients of $\sin\theta$: $R\sin\alpha = 1$ .... [2]

[2] ÷ [1]:

$$\tan\alpha = \frac{1}{3} \Rightarrow \alpha = 18.43^\circ$$
$$R^2 = 3^2 + 1^2 = 10$$
$$R = \sqrt{10}$$
$$3\cos\theta + \sin\theta = \sqrt{10}\cos(\theta - 18.43^\circ)$$

ii $\sqrt{10}\cos(2x - 18.43^\circ) = 2$

$$\cos(2x - 18.43^\circ) = \frac{2}{\sqrt{10}}$$

$2x - 18.43^\circ = 50.7684... \Rightarrow x = 34.6^\circ$

or $2x - 18.43^\circ = 50.7684... + 360 = 410.7684 \Rightarrow 214.6^\circ$

or $2x - 18.43^\circ = 360 - 50.7684... = 309.2316... \Rightarrow 163.8^\circ$

or $2x - 18.43^\circ = 309.2316... + 360 = 669.2316... \Rightarrow 343.8^\circ$

11 **a** $\cos(60^\circ - x) + \cos(300^\circ - x)$

$\equiv \cos 60^\circ \cos x + \sin 60^\circ \sin x + \cos 300^\circ \cos x + \sin 300^\circ \sin x$

$\equiv \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

$\equiv \frac{1}{2}\cos x + \frac{1}{2}\cos x$

$\equiv \cos x$

b **i** From part **a**:

$\cos 15^\circ + \cos 255^\circ$

$= \cos(60^\circ - 45^\circ) + \cos(300^\circ - 45^\circ)$

$= \cos 45$

$= \frac{\sqrt{2}}{2}$

ii

$$\cos x = \frac{1}{4}\operatorname{cosec} x$$

$$\cos x = \frac{1}{4\sin x}$$

$$2\sin x\cos x = \frac{1}{2}$$

$$\sin 2x = \frac{1}{2}$$

$2x = 30^\circ \Rightarrow x = 15^\circ$ or $2x = 180^\circ - 30^\circ = 150^\circ \Rightarrow x = 75^\circ$

12 **a** $\frac{2\sin 2\theta - 3\cos 2\theta + 3}{\sin\theta}$

$\equiv \frac{2(2\sin\theta\cos\theta) - 3(1 - 2\sin^2\theta) + 3}{\sin\theta}$

$\equiv \frac{4\sin\theta\cos\theta - 3 + 6\sin^2\theta + 3}{\sin\theta}$

$\equiv \frac{4\sin\theta\cos\theta + 6\sin^2\theta}{\sin\theta}$

$\equiv 4\cos\theta + 6\sin\theta$

> In worked solution **a**, for $\cos\ 2\theta$ we chose to use the identity including $\sin^2\theta$. Dividing by $\sin\theta$ still leaves $\sin\theta$ in the final expression.

b $4\cos\theta + 6\sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Equating coefficients of $\cos\theta$: $R\cos\alpha = 4$ .... [1]

Equating coefficients of $\sin\theta$: $R\sin\alpha = 6$ .... [2]

[2] ÷ [1]:

$$\tan\alpha = \frac{3}{2} \Rightarrow \alpha = 0.98$$

$$R^2 = 4^2 + 6^2 = 52$$

$$R = 2\sqrt{13}$$

$4\cos\theta + 6\sin\theta = 2\sqrt{13}\cos(\theta - 0.98)$

c $\left(\frac{2\sin 2\theta - 3\cos 2\theta + 3}{\sin\theta}\right)^2$

$= (4\cos\theta + 6\sin\theta)^2$

$= \left(2\sqrt{13}\cos(\theta - 0.98)\right)^2$

$= 52\cos^2(\theta - 0.98)$

Greatest value $= 52$

13 **i** $4\sin\theta - 6\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Equating coefficients of $\sin\theta$: $R\cos\alpha = 4$ .... [1]

Equating coefficients of $\cos\theta$: $R\sin\alpha = 6$ .... [2]

[2] ÷ [1]:

$$\tan\alpha = \frac{3}{2} \Rightarrow \alpha = 56.31^\circ$$
$$R^2 = 4^2 + 6^2 = 52$$
$$R = 2\sqrt{13}$$
$$4\sin\theta - 6\cos\theta = 2\sqrt{13}\sin(\theta - 56.31^\circ)$$

ii $2\sqrt{13}\sin(\theta - 56.31^\circ) = 3$

$$\sin(\theta - 56.31^\circ) = \frac{3}{2\sqrt{13}}$$
$$\theta - 56.31^\circ = 24.5838...^\circ \Rightarrow \theta = 80.9^\circ$$

or $\theta - 56.31^\circ = 180^\circ - 24.5838...^\circ = 155.4161...^\circ \Rightarrow \theta = 211.7^\circ$

iii $(4\sin\theta - 6\cos\theta)^2 + 8$

$= \left(2\sqrt{13}\sin(\theta - 56.31^\circ)\right)^2 + 8$

$= 52\sin^2(\theta - 56.31^\circ) + 8$

Greatest value $= 52 + 8 = 60$

Least value $= 0 + 8 = 8$

14 **i** $\sin 3\theta$

$\equiv \sin(2\theta + \theta)$

$\equiv \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$\equiv 2\sin\theta\cos\theta\cos\theta + (1 - 2\sin^2\theta)\sin\theta$

$\equiv 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$

$\equiv 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$

$\equiv 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$\equiv 3\sin\theta - 4\sin^3\theta$

ii $x^3 - x + \frac{1}{6}\sqrt{3} = 0$

Letting $x = \frac{2\sin\theta}{\sqrt{3}}$:

$$\left(\frac{2\sin\theta}{\sqrt{3}}\right)^3 - \frac{2\sin\theta}{\sqrt{3}} + \frac{1}{6}\sqrt{3} = 0$$
$$\frac{8\sin^3\theta}{3\sqrt{3}} - \frac{2\sin\theta}{\sqrt{3}} + \frac{1}{6}\sqrt{3} = 0$$
$$16\sin^3\theta - 12\sin\theta + 3 = 0$$
$$-4(3\sin\theta - 4\sin^3\theta) + 3 = 0$$
$$-4\sin 3\theta + 3 = 0$$
$$\sin 3\theta = \frac{3}{4}$$

iii $\sin 3\theta = \frac{3}{4}$

$3\theta = 48.5903^\circ... \Rightarrow \theta = 16.2^\circ \Rightarrow \frac{2\sin\theta}{\sqrt{3}} = 0.322$

or $3\theta = 180^\circ - 48.5903...^\circ = 131.4097...^\circ \Rightarrow \theta = 43.8^\circ \Rightarrow \frac{2\sin\theta}{\sqrt{3}} = 0.799$

or $3\theta = 131.4097^\circ... - 360^\circ \Rightarrow \theta = -76.2^\circ \Rightarrow \frac{2\sin\theta}{\sqrt{3}} = -1.12$

Note that this is a cubic equation, so three different solutions will be needed. Keep adding and subtracting 360° from the values of 3θ that you already have until you get all of the solutions.

15 **a** $\dfrac{1}{\sin(x+30)+\cos(x+60)}$

$\equiv \dfrac{1}{\sin x\cos 30+\cos x\sin 30+\cos x\cos 60-\sin x\sin 60}$

$\equiv \dfrac{1}{\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x+\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x}$

$\equiv \dfrac{1}{\cos x}$

$\equiv \sec x$

b

$$\sec x = 7-\tan^2 x$$
$$\sec x = 7-(\sec^2 x-1)$$
$$\sec^2 x+\sec x-8=0$$

Letting $u=\sec x$:

$$u^2+u-8=0$$
$$u=\frac{-1\pm\sqrt{1^2-4\times 1\times(-8)}}{2\times 1}=\frac{-1\pm\sqrt{33}}{2}$$
$$\sec x=\frac{-1\pm\sqrt{33}}{2}$$
$$\cos x=\frac{2}{-1+\sqrt{33}} \text{ or } \cos x=\frac{2}{-1-\sqrt{33}}$$
$$x=60,\ x=104.5,\ x=255.5,\ x=300$$

16 **a** $\operatorname{cosec}^4 x-\cot^4 x$

$=(\operatorname{cosec}^2 x+\cot^2 x)(\operatorname{cosec}^2 x-\cot^2 x)$

$=(\operatorname{cosec}^2 x+\cot^2 x)(1+\cot^2 x-\cot^2 x)$

$=\operatorname{cosec}^2 x+\cot^2 x$

b

$$\operatorname{cosec}^2 x+\cot^2 x=16-\cot x$$
$$(1+\cot^2 x)+\cot^2 x=16-\cot x$$
$$2\cot^2 x+\cot x-15=0$$

Letting $u=\cot x$:

$$(2u-5)(u+3)=0$$

$u=\frac{5}{2}$ or $u=-3$

$\cot x=\frac{5}{2}$ $\cot x=-3$

$\tan x=\frac{2}{5}$ $\tan x=-\frac{1}{3}$

$x=21.8^\circ$ $x=-18.4^\circ+180^\circ=161.6^\circ$

17 **a** At the points of intersection:

$$1+4\cos 2x=2\cos^2 x-4\sin 2x$$
$$2\cos^2 x-1-4\cos 2x=4\sin 2x$$
$$\cos 2x-4\cos 2x=4\sin 2x$$
$$-3\cos 2x=4\sin 2x$$
$$3\cos 2x+4\sin 2x=0$$

b $3\cos 2x+4\sin 2x=R\sin 2x\cos\alpha+R\cos 2x\sin\alpha$

Equating coefficients of $\sin 2x$: $R\cos\alpha=4$ .... [1]

Equating coefficients of $\cos 2x$: $R\sin\alpha=3$ .... [2]

[2] ÷ [1]:

$$\tan\alpha=\frac{3}{4}\Rightarrow\alpha=36.87^\circ$$
$$R^2=3^2+4^2=25$$
$$R=5$$
$$4\sin 2x+3\cos 2x=5\sin(2x+36.87^\circ)$$

c $5\sin(2x + 36.87^\circ) = 0$

$2x + 36.87^\circ = 0 \Rightarrow x = -18.4$ (out of the required range)

or $2x + 36.87^\circ = 180^\circ \Rightarrow x = 71.6^\circ$

or $2x + 36.87^\circ = 360^\circ \Rightarrow x = 161.6^\circ$

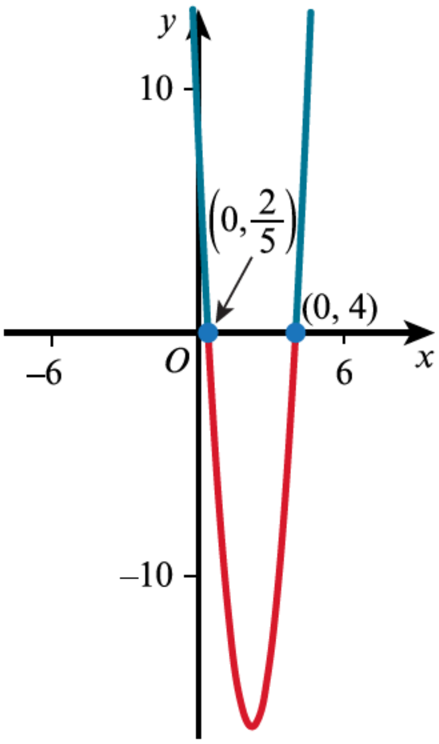
CROSS-TOPIC REVIEW EXERCISE 1

1
$$3|x-1| < |2x+1|$$
$$9(x-1)^2 < (2x+1)^2$$
$$9(x^2-2x+1) < 4x^2+4x+1$$
$$9x^2-18x+9 < 4x^2+4x+1$$
$$5x^2-22x+8 < 0$$

Critical values:

$$5x^2-22x+8=0$$
$$(5x-2)(x-4)=0$$
$$x=\frac{2}{5} \text{ or } x=4$$

Remember that you can also use graphs to solve modulus equations without squaring. A graphical method avoids having to solve quadratics.



$$\frac{2}{5} < x < 4$$

2 $2|3^x-1| = 3^x$

$2(3^x-1)=3^x$

$2(3^x)-2=3^x$

$3^x=2$

$\log 3^x = \log 2$

$x\log 3 = \log 2$

$x=\frac{\log 2}{\log 3} = 0.631$ (to 3 significant figures)

or

$2(3^x-1) = -3^x$

$2(3^x)-2=-3^x$

$3(3^x)=2$

$3^{x+1}=2$

$\log\left(3^{x+1}\right)=\log 2$

$(x+1)\log 3 = \log 2$

$x+1=\frac{\log 2}{\log 3}$

$x=\frac{\log 2}{\log 3}-1=-0.369$ (to 3 significant figures)

Here, it does not matter which logarithm base you use, as long as it is the same throughout each equation.

3 **i**

$$|3x+4| = |3x-11|$$
$$(3x+4)^2 = (3x-11)^2$$
$$9x^2+24x+16 = 9x^2-66x+121$$
$$90x = 105$$
$$x = \frac{105}{90} = \frac{7}{6}$$

ii The equation is the same as in part **i**, but with $x = 2^y$.

$$2^y = \frac{7}{6}$$
$$\log 2^y = \log\frac{7}{6}$$
$$y\log 2 = \log\frac{7}{6}$$
$$y = \frac{\log\frac{7}{6}}{\log 2} = 0.222 \text{ (3 s. f.)}$$

4 **i**

$$2|x-1| = 3|x|$$
$$4(x-1)^2 = 9x^2$$
$$4(x^2-2x+1) = 9x^2$$
$$4x^2-8x+4 = 9x^2$$
$$5x^2+8x-4 = 0$$
$$(5x-2)(x+2) = 0$$
$$x = \frac{2}{5} \text{ or } x = -2$$

ii $5^x = \frac{2}{5}$ or $5^x = -2$

$\log\ 5^x = \log\frac{2}{5}$ (No real solutions for $5^x = -2$ because $5^x > 0$.)

$$x\log 5 = \log\frac{2}{5}$$
$$x = \frac{\log\frac{2}{5}}{\log 5} = -0.569 \text{ (3 significant figures)}$$

If powers of positive numbers are equated to negatives then you must state that there are no real solutions and also include a reason.

5

$$y = A(b^x)$$
$$\ln y = \ln A + \ln b^x$$
$$\ln y = \ln A + x\ln b$$
$$\ln y = (\ln b)x + \ln A$$
$$Y = mX + c$$
$$Y = \ln y$$
$$m = \ln b$$
$$X = x$$
$$c = \ln A$$

$$\text{Gradient} = m = \frac{4.49-2.14}{5-0}$$
$$= \frac{2.35}{5} = 0.47$$
$$\ln b = 0.47$$
$$b = e^{0.47} = 1.6 \ \text{(1 d.p.)}$$
$$c = \ln A = 2.14$$

$A = e^{2.14} = 8.5$ (1 d.p.)

6 **i** $$3^{2x} = 5(3^x) + 14$$
$$(3^x)^2 = 5(3^x) + 14$$
$$(3^x)^2 - 5(3^x) - 14 = 0$$
Letting $u = 3^x$:
$$u^2 - 5u - 14 = 0$$
$$(u-7)(u+2) = 0$$
$u = 7$ or $u = -2$

$3^x = 7$ $3^x = -2$

$3^x = -2$ No real solutions because $3^x > 0$.

$x \log 3 = \log 7$

$$x = \frac{\log 7}{\log 3} = 1.77 \text{ (3 significant figures)}$$

ii $|x| = 1.77$
$x = \pm 1.77$

7 **i** $$x^n y = C$$
$$(1.1)^n(5.2) = C$$
$$(3.2)^n(1.05) = C$$
Dividing:
$$\left(\frac{3.2}{1.1}\right)^n \left(\frac{1.05}{5.2}\right) = 1$$
$$\left(\frac{3.2}{1.1}\right)^n = \frac{5.2}{1.05}$$
$$n \log\left(\frac{3.2}{1.1}\right) = \log\left(\frac{5.2}{1.05}\right)$$
$$n = \frac{\log\left(\frac{5.2}{1.05}\right)}{\log\left(\frac{3.2}{1.1}\right)} = 1.50$$
$$C = 6.00$$

ii $$x^n y = C$$
$$\ln(x^n y) = \ln C$$
$$\ln x^n + \ln y = \ln C$$
$$n \ln x + \ln y = \ln C$$
$$\ln y = -n \ln x + \ln C$$
Which is in the form of the line $Y = mX + K$.

$Y = \ln y$
$m = -n$
$X = \ln x$
$K = \ln C$

> Always set out which variables are playing which roles in the equation of a line.

8 $$3e^x + 8e^{-x} = 14$$
$$3(e^x)^2 + 8 = 14e^x$$
$$3(e^x)^2 - 14(e^x) + 8 = 0$$
$$(3e^x - 2)(e^x - 4) = 0$$
$e^x = \frac{2}{3}$ or $e^x = 4$

$x = \ln\frac{2}{3} = -0.405$ or $x = \ln 4 = 1.39$ (to 3 significant figures)

9

$$\tan(\theta - \phi) = 3$$

$$\frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = 3$$

$$\tan\theta - \tan\phi = 3 + 3\tan\theta\tan\phi$$

But $\tan\theta + \tan\phi = 1$

and $\tan\theta = 1 - \tan\phi$

$$1 - \tan\phi - \tan\phi = 3 + 3(1 - \tan\phi)\tan\phi$$

$$1 - 2\tan\phi = 3 + 3\tan\phi - 3\tan^2\phi$$

$$3\tan^2\phi - 5\tan\phi - 2 = 0$$

$$(3\tan\phi + 1)(\tan\phi - 2) = 0$$

$\tan\phi = -\frac{1}{3}$ or $\tan\phi = 2$

$\tan\theta = 1 - \left(-\frac{1}{3}\right) = \frac{4}{3}$ $\qquad$ $\tan\theta = 1 - 2 = -1$

$\phi = 161.6^\circ$ $\qquad$ $\phi = 63.4^\circ$

$\theta = 53.1^\circ$ $\qquad$ $\theta = 135^\circ$

10 **i**

$$|4x - 1| = |x - 3|$$

$$(4x - 1)^2 = (x - 3)^2$$

$$16x^2 - 8x + 1 = x^2 - 6x + 9$$

$$15x^2 - 2x - 8 = 0$$

$$(5x - 4)(3x + 2) = 0$$

$$x = -\frac{2}{3} \quad \text{or} \quad x = \frac{4}{5}$$

ii This equation is equivalent to the equation in part **i** with $x = 4^y$.

$$4^y = -\frac{2}{3} \quad \text{or} \quad \frac{4}{5}$$

$$4^y > 0$$

$4^y = \frac{4}{5}$ is the only solution.

$$y\ln 4 = \ln\left(\frac{4}{5}\right)$$

$$y = \frac{\ln\left(\frac{4}{5}\right)}{\ln 4} = -0.161 \text{ (to 3 significant figures)}$$

11 **i** $p(x) = 4x^3 + ax^2 + 9x + 9$

$p\left(\frac{1}{2}\right) = 10$ by the remainder theorem.

$$4\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 + 9\left(\frac{1}{2}\right) + 9 = 10$$

$$\frac{1}{2} + \frac{a}{4} + \frac{9}{2} + 9 = 10$$

$$2 + a + 18 + 36 = 40$$

$$a = -16$$

$$p(x) = 4x^3 - 16x^2 + 9x + 9$$

$$p(3) = 4(27) - 16(9) + 27 + 9 = 0$$

$x - 3$ is a factor of $p(x)$ by the factor theorem.

ii Dividing $p(x)$ by $x - 3$ using long division:

$$\begin{array}{r} 4x^2 - 4x - 3 \\ x - 3\overline{)4x^3 - 16x^2 + 9x + 9} \end{array}$$

$$\begin{array}{l}\underline{4x^3 - 12x^2}\\ \quad - 4x^2 + 9x\\ \quad \underline{-4x^2 + 12x}\\ \qquad - 3x + 9\\ \qquad \underline{-3x + 9}\\ \qquad\qquad 0\end{array}$$

$$\begin{aligned}4x^3 - 16x^2 + 9x + 9 &= (x-3)(4x^2 - 4x - 3)\\ &= (x-3)(2x-3)(2x+1) = 0\\ x &= 3, \frac{3}{2}, -\frac{1}{2}\end{aligned}$$

Remember that if the product of two or more brackets is zero, then solutions are found by setting each of the brackets to zero separately.

12 **i** By the remainder theorem $p(-1) = 4$ and $p(3) = 12$.

$p(x) = 2x^3 - 4x^2 + ax + b$

$p(-1) = -2 - 4 - a + b = 4$

$a - b = -10$ .... [1]

$p(3) = 54 - 36 + 3a + b = 12$

$3a + b = -6$ .... [2]

[1] + [2] :

$4a = -16$

$a = -4$

Then from [1]:

$-4 - b = -10$

$b = 6$

ii $p(x) = 2x^3 - 4x^2 - 4x + 6$

Dividing:

$$\begin{array}{r}2x - 4\\ x^2 + 0x - 2\overline{)2x^3 - 4x^2 - 4x + 6}\\ \underline{2x^3 + 0x^2 - 4x}\quad\\ - 4x^2 + 0x + 6\\ \underline{-4x^2 + 0x + 8}\\ -2\end{array}$$

Quotient $= 2x - 4$

Remainder $= -2$

13 **i** $\cos x + 3\sin x = R\cos x\cos\alpha + R\sin x\sin\alpha$

$R\cos\alpha = 1$

$R\sin\alpha = 3$

$R^2 = 1^2 + 3^2 = 10$

$R = \sqrt{10}$

$\tan\alpha = 3$

$\alpha = 71.57^\circ$ (to 2 d.p.)

$\cos x + 3\sin x = \sqrt{10}\cos(x - 71.57^\circ)$

ii Using the result from part **i**:

$\sqrt{10}\cos(2\theta - 71.57^\circ) = 2$

$$\begin{aligned}\cos(2\theta - 71.57^\circ) &= \frac{2}{\sqrt{10}}\\ 2\theta - 71.57^\circ &= 50.7685, -50.7685\\ 2\theta &= 122.338,\ 2\theta = 20.802\\ \theta &= 61.2^\circ,\ \theta = 10.4^\circ\end{aligned}$$

14 **i**

$$
\begin{aligned}
2\ln(4x-5)+\ln(x+1) &= 3\ln 3\\
\ln(4x-5)^2+\ln(x+1) &= \ln 3^3\\
\ln\left[(4x-5)^2(x+1)\right] &= \ln 27\\
(4x-5)^2(x+1) &= 27\\
(16x^2-40x+25)(x+1) &= 27\\
16x^3+16x^2-40x^2-40x+25x+25 &= 27\\
16x^3-24x^2-15x-2 &= 0
\end{aligned}
$$

ii

> When trying to find factors using the factor theorem, remember to try the factors of the constant term first.

$p(x)=16x^3-24x^2-15x-2$

$p(2)=128-96-30-2=0$

$(x-2)$ is a factor of $p(x)$ by the factor theorem.

Dividing:

$$
\begin{array}{r}
16x^2+8x+1\\
x-2\,\overline{)\,16x^3-24x^2-15x-2}\\
\underline{16x^3-32x^2}\qquad\qquad\\
8x^2-15x\qquad\\
\underline{8x^2-16x}\qquad\\
x-2\\
\underline{x-2}\\
0
\end{array}
$$

$$
\begin{aligned}
16x^3-24x^2-15x-2 &= (x-2)(16x^2+8x+1)\\
&= (x-2)(4x+1)^2
\end{aligned}
$$

iii Using the results in parts **i** and **ii**:

$(x-2)(4x+1)^2=0$

$x=2,\ x=-\frac{1}{4}$

But the equation involves the logarithm of $4x-5=-6$ when $x=-\frac{1}{4}$.

So the only solution is $x=2$.

15 **i** $p(x)=8x^3+ax^2+bx-1$

By the factor and remainder theorems:

$p(-1)=0$ and $p\left(-\frac{1}{2}\right)=1$

$$
\begin{aligned}
8(-1)^3+a(-1)^2+b(-1)-1 &= 0\\
-8+a-b-1 &= 0\\
a-b &= 9 \text{ .... [1]}
\end{aligned}
$$

$$
\begin{aligned}
8\left(-\frac{1}{2}\right)^3+a\left(-\frac{1}{2}\right)^2+b\left(-\frac{1}{2}\right)-1 &= 1\\
-1+\frac{a}{4}-\frac{b}{2}-1 &= 1\\
a-2b &= 12 \text{ ...... [2]}
\end{aligned}
$$

[1] − [2]:

$b=-3$

Then from [1]:

$$
\begin{aligned}
a-(-3) &= 9\\
a &= 6
\end{aligned}
$$

$p(-1)=0$ and $p\left(-\frac{1}{2}\right)=1$

$$8(-1)^3 + a(-1)^2 + b(-1) - 1 = 0$$
$$-8 + a - b - 1 = 0$$

$p(x) = 8x^3 + ax^2 + bx - 1$

By the factor and remainders theorems:

$a - b = 9$ .... [1]

$$8\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 + b\left(-\frac{1}{2}\right) - 1 = 1$$
$$-1 + \frac{a}{4} - \frac{b}{2} - 1 = 1$$

$a - 2b = 12$ .... [2]

[1] − [2]:

$b = -3$

Then from [1]:

$$a - (-3) = 9$$
$$a = 6$$

ii $p(x) = 8x^3 + 6x^2 - 3x - 1$

Dividing:

$$\begin{array}{r} 8x^2 - 2x - 1 \\ x+1 \overline{)\, 8x^3 + 6x^2 - 3x - 1} \\ \underline{8x^3 + 8x^2} \qquad\quad \\ -2x^2 - 3x \quad \\ \underline{-2x^2 - 2x} \quad \\ -x - 1 \\ \underline{-x - 1} \\ 0 \end{array}$$

$$8x^3 + 6x^2 - 3x - 1 = (x+1)(8x^2 - 2x - 1)$$
$$= (x+1)(4x+1)(2x-1)$$

Notice that part **ii** of Question **15** does not make any reference to 'solve'. You can, therefore, stop at the point of factorisation.

16 **i** $p(x) = 3x^3 + 2x^2 + ax + b$

By the factor and remainder theorems:

$p(1) = 0$ and $p(2) = 10$

$$3(1)^3 + 2(1)^2 + a(1) + b = 0$$
$$3 + 2 + a + b = 0$$
$$a + b = -5 \text{ .... [1]}$$

$$3(2)^3 + 2(2)^2 + a\,(2) + b = 10$$
$$24 + 8 + 2a + b = 10$$
$$2a + b = -22 \text{ ...... [2]}$$

[2] − [1]:

$a = -17$

Then from [1]:

$$-17 + b = -5$$
$$b = 12$$

ii $p(x) = 3x^3 + 2x^2 - 17x + 12$

Dividing:

$$\begin{array}{r} 3x^2 + 5x - 12 \\ x-1 \overline{)\, 3x^3 + 2x^2 - 17x + 12} \end{array}$$

$$\underline{3x^3 - 3x^2}$$
$$5x^2 - 17x$$
$$\underline{5x^2 - 5x}$$
$$-\,12x + 12$$
$$\underline{-12x + 12}$$
$$0$$

$$\begin{aligned} 3x^3 + 2x^2 - 17x + 12 &= (x-1)(3x^2 + 5x - 12) \\ &= (x-1)(3x-4)(x+3) = 0 \\ x &= 1, \frac{4}{3}, -3 \end{aligned}$$

17 **i** $p(x) = x^3 + ax^2 + bx + 8$

By the remainder theorem:

$p(3) = 14$ and $p(-2) = 24$

$$\begin{aligned} 27 + 9a + 3b + 8 &= 14 \\ 9a + 3b &= -21 \\ 3a + b &= -7 \text{ .... [1]} \end{aligned}$$

$$\begin{aligned} -8 + 4a - 2b + 8 &= 24 \\ 4a - 2b &= 24 \\ 2a - b &= 12 \text{ .... [2]} \end{aligned}$$

[1] + [2]:

$$\begin{aligned} 5a &= 5 \\ a &= 1 \end{aligned}$$

Then from [1]:

$$\begin{aligned} 3 + b &= -7 \\ b &= -10 \end{aligned}$$

ii $p(x) = x^3 + x^2 - 10x + 8$

Dividing:

$$x - 1$$
$$x^2 + 2x - 8 \overline{\smash{)}\, x^3 + x^2 - 10x + 8}$$
$$\underline{x^3 + 2x^2 - 8x}$$
$$-\,x^2 - 2x + 8$$
$$\underline{-x^2 - 2x + 8}$$
$$0$$

$$\begin{aligned} x^3 + x^2 - 10x + 8 &= \left(x^2 + 2x - 8\right)(x-1) \\ &= (x+4)(x-2)(x-1) \\ x &= -4, 2, 1 \end{aligned}$$

18 **i** $p(x) = 5x^3 + ax^2 + b$

By the factor theorem:

$p(-2) = 0$ and $p(-3) = 0$

$$\begin{aligned} -40 + 4a + b &= 0 \\ 4a + b &= 40 \text{ .... [1]} \end{aligned}$$

$$\begin{aligned} -135 + 9a + b &= 0 \\ 9a + b &= 135 \text{ ..... [2]} \end{aligned}$$

[2] − [1]:

$$\begin{aligned} 5a &= 95 \\ a &= 19 \end{aligned}$$

Then from [1]:

$$\begin{aligned} 76 + b &= 40 \\ b &= -36 \end{aligned}$$

ii

> Remember that if two different linear polynomials are a factor of another polynomial then so is

their product.

$p(x) = 5x^3 + 19x^2 - 36$

$x + 2$ and $x + 3$ are both factors of $p(x)$

So

$(x + 2)(x + 3) = x^2 + 5x + 6$ is also a factor of $p(x)$.

$$\begin{array}{r}
5x - 6 \\
x^2 + 5x + 6 \overline{\smash{)}5x^3 + 19x^2 + 0x - 36} \\
\underline{5x^3 + 25x^2 + 30x} \\
-6x^2 - 30x - 36 \\
\underline{-6x^2 - 30x - 36} \\
0
\end{array}$$

$$\begin{aligned} 5x^3 + 19x^2 + 0x - 36 &= (x^2 + 5x + 6)(5x - 6) \\ &= (x + 3)(x + 2)(5x - 6) \\ x &= -3, -2, \frac{6}{5} \end{aligned}$$

In the new equation

$x = 5^y$

$5^y > 0$

$5^y = \frac{6}{5}$ is the only solution

$y \ln 5 = \ln\left(\frac{6}{5}\right)$

$y = \frac{\ln\left(\frac{6}{5}\right)}{\ln 5} = 0.113$ (to 3 significant figures)

19 **i** $p(x) = x^4 + x^3 + 3x^2 + 12x + 6$

$$\begin{array}{r}
x^2 + 2x + 1 \\
x^2 - x + 4 \overline{\smash{)}x^4 + x^3 + 3x^2 + 12x + 6} \\
\underline{x^4 - x^3 + 4x^2} \\
2x^3 - x^2 + 12x \\
\underline{2x^3 - 2x^2 + 8x} \\
x^2 + 4x + 6 \\
\underline{x^2 - x + 4} \\
5x + 2
\end{array}$$

Quotient $= x^2 + 2x + 1$

Remainder $= 5x + 2$

ii

$$\begin{array}{r}
x^2 + 2x + 1 \\
x^2 - x + 4 \overline{\smash{)}x^4 + x^3 + 3x^2 + px + q} \\
\underline{x^4 - x^3 + 4x^2} \\
2x^3 - x^2 + px \\
\underline{2x^3 - 2x^2 + 8x} \\
x^2 + (p - 8)x + q \\
\underline{x^2 - x + 4} \\
(p - 8 + 1)x + q - 4
\end{array}$$

Remainder $= (p - 7)x + q - 4 = 0$

$p = 7$

$q = 4$

iii Dividing:

$$\begin{array}{r}
x^2 + 2x + 1 \\
x^2 - x + 4 \overline{\smash{)}x^4 + x^3 + 3x^2 + 7x + 4}
\end{array}$$

$$\begin{array}{l} \underline{x^4 - x^3 + 4x^2} \\ \quad 2x^3 - x^2 + 7x \\ \quad \underline{2x^3 - 2x^2 + 8x} \\ \qquad x^2 - x + 4 \\ \qquad \underline{x^2 - x + 4} \\ \qquad\qquad 0 \end{array}$$

$$\begin{aligned} x^4 + x^3 + 3x^2 + 7x + 4 &= (x^2 - x + 4)(x^2 + 2x + 1) \\ &= (x+1)^2(x^2 - x + 4) \end{aligned}$$

Note that $x^2 - x + 4 \neq 0$ because $b^2 - 4ac = 1 - 4(1)(4) < 0$

So

$(x+1)^2 = 0$ is the only possibility.

$x = -1$ is the only solution.

20 **i**

$$\begin{aligned} 2\tan^2\alpha + \sec^2\alpha &= 5 - 4\tan\alpha \\ \text{but } \sec^2\alpha &\equiv 1 + \tan^2\alpha \\ 2\tan^2\alpha + 1 + \tan^2\alpha &= 5 - 4\tan\alpha \\ 3\tan^2\alpha + 4\tan\alpha - 4 &= 0 \\ (3\tan\alpha - 2)(\tan\alpha + 2) &= 0 \end{aligned}$$

$\tan\alpha = \frac{2}{3}$ or $\tan\alpha = -2$

α is acute, so $\tan\alpha > 0$

$\tan\alpha = \frac{2}{3}$

ii

$$\begin{aligned} \cot(\alpha + \beta) &= 6 \\ \frac{1}{\tan(\alpha + \beta)} &= 6 \\ \tan(\alpha + \beta) &= \frac{1}{6} \\ \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} &= \frac{1}{6} \\ \frac{\frac{2}{3} + \tan\beta}{1 - \frac{2}{3}\tan\beta} &= \frac{1}{6} \\ 4 + 6\tan\beta &= 1 - \frac{2}{3}\tan\beta \\ 12 + 18\tan\beta &= 3 - 2\tan\beta \\ 20\tan\beta &= -9 \\ \tan\beta &= -\frac{9}{20} \end{aligned}$$

21 **i**

As a minimum you should show the expansion of $R\cos(\theta + \alpha)$.

$$\begin{aligned} 5\sin 2\theta + 2\cos 2\theta &= R\sin 2\theta\cos\alpha + R\cos 2\theta\sin\alpha \\ R\cos\alpha &= 5 \\ R\sin\alpha &= 2 \\ R^2 &= 5^2 + 2^2 = 29 \\ R &= \sqrt{29} \\ \tan\alpha &= \frac{2}{5} \end{aligned}$$

$\alpha = 21.80^\circ$ (to 2 d.p.)

$5\sin 2\theta + 2\cos 2\theta = \sqrt{29}\sin(2\theta + 21.80^\circ)$

ii Using the result from part **i**:

$$\sqrt{29}\sin(2\theta+21.80^\circ)=4$$

$$\sin(2\theta+21.80^\circ)=\frac{4}{\sqrt{29}}$$

$$2\theta+21.80^\circ=47.96889^\circ,\ 132.03111^\circ,\ 407.96889^\circ,\ 492.03111^\circ$$

$$\theta=13.1^\circ,\ 55.1^\circ,\ 193.1^\circ,\ 235.1^\circ$$

iii

$$10\sin 2\theta+4\cos 2\theta=2\left(\sqrt{29}\sin(2\theta+21.80^\circ)\right)$$

$$(10\sin 2\theta+4\cos 2\theta)^2=4\left(\sqrt{29}\sin(2\theta+21.80^\circ)\right)^2$$

$$=116\sin^2(2\theta+21.80^\circ)$$

$$\frac{1}{(10\sin 2\theta+4\cos 2\theta)^2}=\frac{1}{116\sin^2(2\theta+21.80^\circ)}$$

Least value when the denominator is at a maximum.

$$\sin^2(2\theta+21.80^\circ)=1$$

$$\max\frac{1}{(10\sin 2\theta+4\cos 2\theta)^2}=\frac{1}{116}$$

22 **i** $p(x)=ax^3+3x^2+bx+12$

By the factor and remainder theorems:

$P(-3)=0$ and $p(-2)=18$

$$-27a+27-3b+12=0$$

$$27a+3b=39$$

$$9a+b=13\ \text{.........}\ [1]$$

$$-8a+12-2b+12=18$$

$$8a+2b=6$$

$$4a+b=3\ \text{............}\ [2]$$

[1] − [2]:

$$5a=10$$

$$a=2$$

Then from [1]:

$$18+b=13$$

$$b=-5$$

ii **a** $p(x)=2x^3+3x^2-5x+12$

Dividing:

$$\begin{array}{r}
2x^2-3x+4\\
x+3\overline{\big)\,2x^3+3x^2-5x+12}\\
\underline{2x^3+6x^2}\qquad\qquad\\
-3x^2-5x\qquad\\
\underline{-3x^2-9x}\qquad\\
4x+12\\
\underline{4x+12}\\
0
\end{array}$$

$$2x^3+3x^2-5x+12=(x+3)(2x^2-3x+4)$$

Note that, for $2x^2-3x+4$,

$$b^2-4ac=9-4(1)(4)<0$$

$$2x^2-3x+4\neq 0$$

$x=-3$ is the only real solution.

b $\sec y=-3$

$$\cos y=-\frac{1}{3}$$

$$y=\pm 109.5^\circ$$

Chapter 4

Differentiation

EXERCISE 4A

1 **b**

$$y = 5x(2x+1)^3$$
$$\begin{aligned}\frac{dy}{dx} &= 5x\frac{d}{dx}\left[(2x+1)^3\right] + (2x+1)^3\frac{d}{dx}[5x]\\ &= 5x[3(2x+1)^2(2)] + (2x+1)^3[5]\\ &= 30x(2x+1)^2 + 5(2x+1)^3\\ &= (2x+1)^2[30x+5(2x+1)]\\ &= (2x+1)^2(30x+10x+5)\\ &= (2x+1)^2(40x+5)\\ &= 5(2x+1)^2(8x+1)\end{aligned}$$

h

$$y = (2x-1)^5(3x+4)^4$$
$$\begin{aligned}\frac{dy}{dx} &= (2x-1)^5\frac{d}{dx}\left[(3x+4)^4\right] + (3x+4)^4\frac{d}{dx}\left[(2x-1)^5\right]\\ &= (2x-1)^5[4(3x+4)^3(3)] + (3x+4)^4[5(2x-1)^4(2)]\\ &= (2x-1)^5[12(3x+4)^3] + (3x+4)^4[10(2x-1)^4]\\ &= 12(2x-1)^5(3x+4)^3 + 10(3x+4)^4(2x-1)^4\\ &= 2(2x-1)^4(3x+4)^3[6(2x-1)+5(3x+4)]\\ &= 2(2x-1)^4(3x+4)^3(12x-6+15x+20)\\ &= 2(2x-1)^4(3x+4)^3(27x+14)\end{aligned}$$

2

$$y = x^2(x+4)^{\frac{1}{2}}$$
$$\begin{aligned}\frac{dy}{dx} &= x^2\frac{d}{dx}[(x+4)^{\frac{1}{2}}] + (x+4)^{\frac{1}{2}}\frac{d}{dx}[x^2]\\ &= x^2\left[\frac{1}{2}(x+4)^{-\frac{1}{2}}\right] + (x+4)^{\frac{1}{2}}[2x]\\ &= \frac{x^2}{2\sqrt{(x+4)}} + 2x\sqrt{x+4}\\ &= \frac{x^2+4x(x+4)}{2\sqrt{(x+4)}}\\ &= \frac{5x^2+16x}{2\sqrt{(x+4)}}\end{aligned}$$

When $x=-3$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{5(-3)^2+16(-3)}{2\sqrt{(-3)+4}}\\ &= \frac{45-48}{2\sqrt{1}} = -\frac{3}{2} = -1.5\end{aligned}$$

3

$$y = (2-x)^3(x+1)^4$$
$$\begin{aligned}\frac{dy}{dx} &= (2-x)^3[4(x+1)^3] + (x+1)^4[3(2-x)^2(-1)]\\ &= 4(2-x)^3(x+1)^3 - 3(x+1)^4(2-x)^2\end{aligned}$$

When $x=1$:

$$\frac{dy}{dx} = 4(1)^3(2)^3 - 3(2)^4(1)^2$$
$$= 32 - 48$$
$$= -16$$

When $x = 1$, $y = (2-1)^3(1+1)^4 = 16$

Equation of the tangent is:

$$y - 16 = -16(x-1)$$
$$y - 16 = -16x + 16$$
$$16x + y = 32$$

Question **3** did not ask for a particular form to be used. When this happens it is best to write the equation of a straight line in the form $ax + by = c$.

4

$$y = (x+2)(x-1)^3$$
$$\frac{dy}{dx} = (x+2)[3(x-1)^2] + (x-1)^3[(1)]$$
$$= 3(x+2)(x-1)^2 + (x-1)^3$$

At the point where the curve meets the y-axis, $x = 0$.

When $x = 0$:

$$\frac{dy}{dx} = 3(2)(-1)^2 + (-1)^3$$
$$= 6 - 1$$
$$= 5$$

5

$$y = (3-x)^3(x+1)^2$$
$$\frac{dy}{dx} = (3-x)^3[2(x+1)] + (x+1)^2[3(3-x)^2(-1)]$$
$$= 2(3-x)^3(x+1) - 3(x+1)^2(3-x)^2$$
$$= (x+1)(3-x)^2[2(3-x) - 3(x+1)]$$
$$= (x+1)(3-x)^2(6 - 2x - 3x - 3)$$
$$= (x+1)(3-x)^2(3-5x)$$

When $\frac{dy}{dx} = 0$:

$x = -1, 3, \frac{3}{5}$

When you are asked for turning points it is usually necessary to set a polynomial equal to zero and solve the resulting equation. You will need to factorise fully, which is why it is good to practice factorising whenever you differentiate.

6

$$y = (x+2)\sqrt{1-2x} = (x+2)(1-2x)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = (x+2)\left[\frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2)\right] + [(1)](1-2x)^{\frac{1}{2}}$$
$$= -(x+2)(1-2x)^{-\frac{1}{2}} + (1-2x)^{\frac{1}{2}}$$
$$= -\frac{x+2}{\sqrt{1-2x}} + \sqrt{1-2x}$$
$$= \frac{-(x+2) + (1-2x)}{\sqrt{1-2x}}$$
$$= \frac{-x - 2 + 1 - 2x}{\sqrt{1-2x}}$$
$$= \frac{-1-3x}{\sqrt{1-2x}}$$

When $\frac{dy}{dx} = 0$:

$$-1-3x=0$$
$$3x=-1$$
$$x=-\frac{1}{3}$$

Remember that a fraction is zero when the numerator is zero.

7 **a** $y=(x-1)^2(5-2x)+3$

$$\frac{dy}{dx}=(x-1)^2(-2)+(5-2x)[2(x-1)]$$
$$=-2(x-1)^2+2(5-2x)(x-1)$$
$$=(x-1)[2(5-2x)-2(x-1)]$$
$$=(x-1)(10-4x-2x+2)$$
$$=(x-1)(12-6x)$$
$$=6(x-1)(2-x)$$

The stationary points are when $\frac{dy}{dx}=0$.

Stationary points at $x=1$ and $x=2$.

When $x=1$:

$y=(0)^2(5-2)+3=3$

Stationary point at $(1, 3)$.

When $x=2$:

$y=(1)^2(5-4)+3=4$

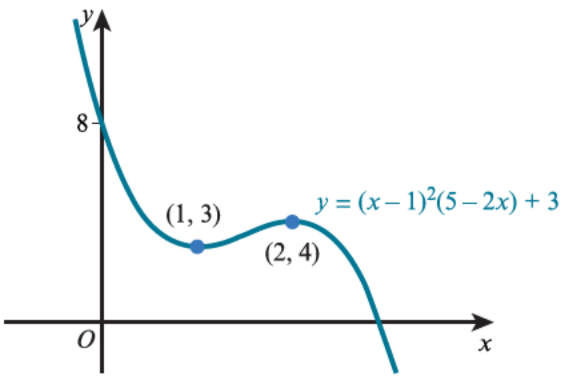
Stationary point at $(2, 4)$.

When $x=0$:

$y=(-1)^2(5-0)+3=8$

The curve meets the y-axis at $y=8$.

Expansion of the brackets will show that the coefficient of x^3 is negative, so the graph will be of the shape shown in the diagram:



b From part **a**, the stationary points have coordinates $A\,(1, 3)$ and $B\,(2, 4)$.

Equation of the line through A and B:

$$\frac{y-3}{x-1}=\frac{4-3}{2-1}$$
$$\frac{y-3}{x-1}=\frac{1}{1}$$
$$y-3=x-1$$
$$x-y=-2$$

$x=0$ gives $0-y=-2$

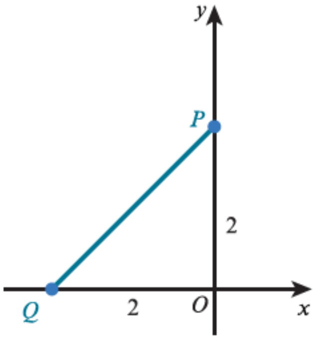
$y=2$

$y=0$ gives $x-0=-2$

$x=-2$

$P(0,2)\quad Q(-2,0)$

The right-angled triangle POQ is as shown in this diagram:



From the diagram:

Area of triangle $POQ = 2 \times 2 \times \frac{1}{2} = 2$

Always draw a diagram for situations like this. It will make it much easier to identify which triangle you need to think about.

EXERCISE 4B

1 **c** $y = \frac{x^2 - 3}{2x - 1}$

$$\frac{dy}{dx} = \frac{(2x-1)\frac{d}{dx}[(x^2-3)] - (x^2-3)\frac{d}{dx}[(2x-1)]}{(2x-1)^2}$$

$$= \frac{(2x-1)(2x) - (x^2-3)(2)}{(2x-1)^2}$$

$$= \frac{4x^2 - 2x - 2x^2 + 6}{(2x-1)^2}$$

$$= \frac{2x^2 - 2x + 6}{(2x-1)^2}$$

$$= \frac{2(x^2 - x + 3)}{(2x-1)^2}$$

You do not need to include the step with terms containing '$\frac{d}{dx}$'. They are shown here so that you can see what is happening. In the worked solutions for later questions the relevant parts are simply differentiated without showing these terms.

f $y = \frac{5x^4}{(x^2-1)^2}$

$$\frac{dy}{dx} = \frac{(x^2-1)^2\frac{d}{dx}[5x^4] - 5x^4\frac{d}{dx}[(x^2-1)^2]}{(x^2-1)^4}$$

$$= \frac{(x^2-1)^2(20x^3) - 5x^4[2(x^2-1)(2x)]}{(x^2-1)^4}$$

$$= \frac{20x^3(x^2-1)^2 - 20x^5(x^2-1)}{(x^2-1)^4}$$

$$= \frac{20x^3(x^2-1)[(x^2-1) - x^2]}{(x^2-1)^4}$$

$$= \frac{20x^3(x^2-1)(-1)}{(x^2-1)^4}$$

$$= -\frac{20x^3}{(x^2-1)^3}$$

2 $y = \frac{x-5}{x+4}$

$$\frac{dy}{dx} = \frac{(x+4)(1) - (x-5)(1)}{(x+4)^2}$$

$$= \frac{x + 4 - x + 5}{(x+4)^2}$$

$$= \frac{9}{(x+4)^2}$$

At $x = 2$:

$$\frac{dy}{dx} = \frac{9}{(2+4)^2}$$

$$= \frac{9}{6^2}$$

$$= \frac{9}{36}$$

$$= \frac{1}{4}$$

3 Whenever you are asked to find the points at which a curve is parallel to a given line, you simply need to find out what the gradient of that line is and set it equal to $\frac{\mathrm{d}y}{\mathrm{d}x}$.

The curve will be parallel to the x-axis when the gradient is zero.

$$y = \frac{(x-1)^2}{2x+5}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(2x+5)[2(x-1)] - (x-1)^2(2)}{(2x+5)^2}$$

$$= \frac{2(2x+5)(x-1) - 2(x-1)^2}{(2x+5)^2}$$

$$= \frac{2(2x^2+3x-5) - 2(x^2-2x+1)}{(2x+5)^2}$$

$$= \frac{4x^2+6x-10-2x^2+4x-2}{(2x+5)^2}$$

$$= \frac{2x^2+10x-12}{(2x+5)^2}$$

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$\frac{2x^2+10x-12}{(2x+5)^2} = 0$$

$$2x^2+10x-12 = 0$$

$$x^2+5x-6 = 0$$

$$(x+6)(x-1) = 0$$

$x = 1$ or $x = -6$

$y = 0$ $\quad y = \frac{(-6-1)^2}{2(-6)+5} = \frac{49}{-7} = -7$

The points are $(1, 0)$ and $(-6, -7)$.

4

$$y = \frac{1-2x}{x-5}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-5)(-2) - (1-2x)(1)}{(x-5)^2}$$

$$= \frac{-2x+10-1+2x}{(x-5)^2}$$

$$= \frac{9}{(x-5)^2}$$

Gradient is 1 when $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$:

$$\frac{9}{(x-5)^2} = 1$$

$$(x-5)^2 = 9$$

$$x - 5 = \pm 3$$

$$x = 5 \pm 3$$

$x = 2$ or $x = 8$

$y = \frac{1-2(2)}{(2)-5} = 1$ $\quad y = \frac{1-2(8)}{(8)-5} = -5$

The points are (2, 1) and (8, −5).

5 $$y = \frac{x-4}{2x+1}$$

$$\frac{dy}{dx} = \frac{(2x+1)(1)-(x-4)(2)}{(2x+1)^2}$$

$$= \frac{2x+1-2x+8}{(2x+1)^2}$$

$$= \frac{9}{(2x+1)^2}$$

Curve crosses the y-axis when $x = 0$.

When $x = 0$:

$$\frac{dy}{dx} = \frac{9}{(2(0)+1)^2} = 9$$

At this point, $y = \frac{0-4}{0+1} = -4 = y$-intercept

Equation of the tangent is:

$y = 9x - 4$

6 **b** $$y = \frac{x-1}{\sqrt{2x+3}} = \frac{x-1}{(2x+3)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{(2x+3)^{\frac{1}{2}}(1)-(x-1)\left[\frac{1}{2}(2x+3)^{-\frac{1}{2}}(2)\right]}{2x+3}$$

$$= \frac{\sqrt{2x+3}-\frac{(x-1)}{\sqrt{2x+3}}}{2x+3}$$

$$= \frac{(2x+3)-(x-1)}{(2x+3)\sqrt{2x+3}}$$

$$= \frac{x+4}{(2x+3)^{\frac{3}{2}}}$$

c $$y = \frac{3-x^2}{\sqrt{x^2-1}} = \frac{3-x^2}{(x^2-1)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{(x^2-1)^{\frac{1}{2}}(-2x)-(3-x^2)\left[\frac{1}{2}(x^2-1)^{-\frac{1}{2}}(2x)\right]}{x^2-1}$$

$$= \frac{-2x\sqrt{x^2-1}-\frac{2x(3-x^2)}{2\sqrt{x^2-1}}}{x^2-1}$$

$$= \frac{-4x(x^2-1)-2x(3-x^2)}{2(x^2-1)^{\frac{3}{2}}}$$

$$= \frac{-4x^3+4x-6x+2x^3}{2(x^2-1)^{\frac{3}{2}}}$$

$$= \frac{-2x-2x^3}{2(x^2-1)^{\frac{3}{2}}}$$

$$= \frac{-x(1+x^2)}{(x^2-1)^{\frac{3}{2}}}$$

Try to use different shaped brackets when writing brackets inside other brackets. In the worked solution for Question **7**, square brackets are used to contain terms with round brackets.

7 $y = \frac{x+1}{\sqrt{x-1}} = \frac{x+1}{(x-1)^{\frac{1}{2}}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-1)^{\frac{1}{2}}(1) - (x+1)\left[\frac{1}{2}(x-1)^{-\frac{1}{2}}\right]}{x-1}$$

$$= \frac{\sqrt{x-1} - \frac{x+1}{2\sqrt{x-1}}}{x-1}$$

$$= \frac{2(x-1) - (x+1)}{2(x-1)^{\frac{3}{2}}}$$

$$= \frac{x-3}{2(x-1)^{\frac{3}{2}}}$$

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$\frac{x-3}{2(x-1)^{\frac{3}{2}}} = 0$$

$$x - 3 = 0$$

$$x = 3$$

8 $y = \frac{x^2+1}{\sqrt{x+2}} = \frac{x^2+1}{(x+2)^{\frac{1}{2}}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+2)^{\frac{1}{2}}(2x) - (x^2+1)\left[\frac{1}{2}(x+2)^{-\frac{1}{2}}\right]}{x+2}$$

$$= \frac{2x\sqrt{x+2} - \frac{(x^2+1)}{2\sqrt{x+2}}}{x+2}$$

$$= \frac{4x(x+2) - (x^2+1)}{2(x+2)^{\frac{3}{2}}}$$

$$= \frac{4x^2 + 8x - x^2 - 1}{2(x+2)^{\frac{3}{2}}}$$

$$= \frac{3x^2 + 8x - 1}{2(x+2)^{\frac{3}{2}}}$$

When $x = -1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3(-1)^2 + 8(-1) - 1}{2((-1)+2)^{\frac{3}{2}}}$$

$$= \frac{3 - 8 - 1}{2} = -3$$

Gradient of the normal $= -\frac{1}{(-3)} = \frac{1}{3}$

Equation of the normal is:

$$y - 2 = \frac{1}{3}(x - -1)$$

$$3y - 6 = x + 1$$

$$3y = x + 7$$

$$x - 3y + 7 = 0$$

> Remember that the normal to a curve at a given point is the line perpendicular to the tangent at the same point.

9 **a** $2x - 2y = 5$

$$2y = 2x - 5$$

$y = \frac{2x-5}{2}$ .... [1]

$2x^2y - x^2 - 26y - 35 = 0$ .... [2]

Substituting [1] into [2]:

$$2x^2\left(\frac{2x-5}{2}\right) - x^2 - 26\left(\frac{2x-5}{2}\right) - 35 = 0$$
$$x^2(2x-5) - x^2 - 13(2x-5) - 35 = 0$$
$$2x^3 - 5x^2 - x^2 - 26x + 65 - 35 = 0$$
$$2x^3 - 6x^2 - 26x + 30 = 0$$
$$x^3 - 3x^2 - 13x + 15 = 0$$

Note that $(1)^3 - 3(1)^2 - 13(1) + 15 = 1 - 3 - 13 + 15 = 0$

$x - 1$ is a factor of $x^3 - 3x^2 - 13x + 15$ by the factor theorem.

Dividing:

$$\begin{array}{r|l} & x^2 - 2x - 15 \\ x - 1 & x^3 - 3x^2 - 13x + 15 \\ & \underline{x^3 - x^2} \\ & -2x^2 - 13x \\ & \underline{-2x^2 + 2x} \\ & -15x + 15 \\ & \underline{-15x + 15} \\ & 0 \end{array}$$

$(x-1)(x^2 - 2x - 15) = 0$

$(x-1)(x-5)(x+3) = 0$

$x = 1$ or $x = -3$ or $x = 5$

b Making y the subject of the equation of the curve:

$$y(2x^2 - 26) - x^2 - 35 = 0$$
$$y(2x^2 - 26) = x^2 + 35$$
$$y = \frac{x^2 + 35}{2x^2 - 26}$$

$$\frac{dy}{dx} = \frac{(2x^2 - 26)(2x) - (x^2 + 35)(4x)}{(2x^2 - 26)^2}$$
$$= \frac{4x^3 - 52x - 4x^3 - 140x}{(2x^2 - 26)^2}$$
$$= \frac{-192x}{2^2(x^2 - 13)^2}$$
$$= -\frac{48x}{(x^2 - 13)^2}$$

$x = 1$ gives $\frac{dy}{dx} = -\frac{48(1)}{((1)^2 - 13)^2} = -\frac{48}{144} = -\frac{1}{3}$

$x = -3$ gives $\frac{dy}{dx} = -\frac{48(-3)}{((-3)^2 - 13)^2} = \frac{-144}{-16} = 9$

$x = 5$ gives $\frac{dy}{dx} = -\frac{48(5)}{((5)^2 - 13)^2} = -\frac{240}{144} = -\frac{5}{3}$

EXERCISE 4C

1

These worked solutions show where the 'inside' function has been differentiated. You don't need to show this in your solutions; instead you can simply differentiate the 'inside' function in your head and multiply by the answer.

e $y = 4e^{\frac{x}{2}}$

$$\frac{dy}{dx} = 4e^{\frac{x}{2}} \times \frac{d}{dx}\left[\frac{x}{2}\right]$$

$$\frac{dy}{dx} = 4e^{\frac{x}{2}} \times \frac{1}{2} = 2e^{\frac{x}{2}}$$

k $y = \dfrac{3e^{2x} + e^{-2x}}{2}$

$$\frac{dy}{dx} = \frac{1}{2}\left[3e^{2x}\frac{d}{dx}(2x) + e^{-2x}\frac{d}{dx}(-2x)\right]$$

$$= \frac{1}{2}[3e^{2x} \times 2 + e^{-2x} \times (-2)]$$

$$= \frac{6e^{2x} - 2e^{-2x}}{2}$$

$$= 3e^{2x} - e^{-2x}$$

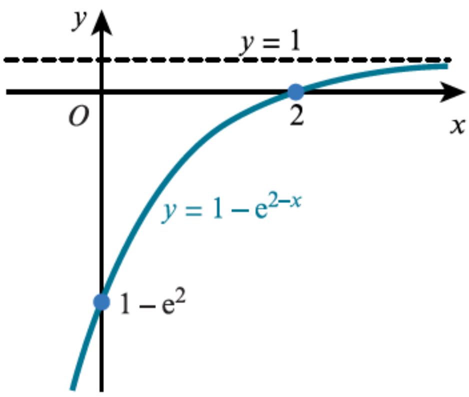
2 **a** Curve crosses the x-axis when:

$$y = 0$$
$$1 - e^{2-x} = 0$$
$$e^{2-x} = 1$$
$$2 - x = 0$$
$$x = 2$$

Curve crosses the y-axis when:

$$x = 0$$
$$y = 1 - e^{2-0}$$
$$= 1 - e^2$$

As x increases, $y = 1 - e^{2-x} = 1 - \dfrac{e^2}{e^x}$ gets closer and closer to 1 because $\dfrac{e^2}{e^x}$ gets closer and closer to zero.



b From part **a**:

$y = 0$ gives $x = 2$

$$\frac{dy}{dx} = 0 - e^{2-x}(-1)$$
$$= e^{2-x}$$

When $x = 2$:

$$\frac{dy}{dx} = e^{2-2} = e^0 = 1$$

Gradient of the normal $= -\dfrac{1}{1} = -1$

Equation of the normal is:

$y - 0 = -1(x - 2)$

$y = -x + 2$

You can also use the $y = mx + c$ form of the equation. There is no set method, but always take care to check what final form of the answer is required.

3

$$m = 300\mathrm{e}^{-0.00012t}$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = 300\mathrm{e}^{-0.00012t} \times (-0.00012)$$

$$= -0.036\mathrm{e}^{-0.00012t}$$

When $t = 2000$:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -0.036\mathrm{e}^{-0.00012 \times 2000} = -0.0283$$

Rate of decrease $= 0.0283$ grams per year.

4 **c**

$$y = 5x\,\mathrm{e}^{-2x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5x\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{e}^{-2x}] + \mathrm{e}^{-2x}\frac{\mathrm{d}}{\mathrm{d}x}[5x]$$

$$= 5x\,\mathrm{e}^{-2x}(-2) + \mathrm{e}^{-2x}(5)$$

$$= (5 - 10x)\,\mathrm{e}^{-2x}$$

g

$$y = \frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\mathrm{e}^x + 2)\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{e}^x - 1] - (\mathrm{e}^x - 1)\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{e}^x + 2]}{(\mathrm{e}^x + 2)^2}$$

$$= \frac{(\mathrm{e}^x + 2)(\mathrm{e}^x) - (\mathrm{e}^x - 1)(\mathrm{e}^x)}{(\mathrm{e}^x + 2)^2}$$

$$= \frac{\mathrm{e}^{2x} + 2\mathrm{e}^x - \mathrm{e}^{2x} + \mathrm{e}^x}{(\mathrm{e}^x + 2)^2}$$

$$= \frac{3\mathrm{e}^x}{(\mathrm{e}^x + 2)^2}$$

5

To answer Question **5**, you could also use the quotient rule. Generally, though, if the numerator of the fraction is just a constant it is easier to use the negative power as shown here.

$$y = \frac{8}{5 + \mathrm{e}^{2x}} = 8(5 + \mathrm{e}^{2x})^{-1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -8(5 + \mathrm{e}^{2x})^{-2}(2\mathrm{e}^{2x})$$

$$= \frac{-16\mathrm{e}^{2x}}{(5 + \mathrm{e}^{2x})^2}$$

When $x = 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-16\,\mathrm{e}^0}{(5 + \mathrm{e}^0)^2}$$

$$= \frac{-16}{6^2}$$

$$= -\frac{16}{36}$$

$$= -\frac{4}{9}$$

6

$$y = x\mathrm{e}^x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^x + \mathrm{e}^x(1) = x\mathrm{e}^x + \mathrm{e}^x = (x + 1)\mathrm{e}^x$$

Stationary point when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$(x + 1)\mathrm{e}^x = 0$

$e^x > 0$, so $x + 1 = 0$

$x = -1$

$y = xe^x = (-1)e^{-1} = -\frac{1}{e}$

Coordinates of stationary point are:

$\left(-1, -\frac{1}{e}\right)$

7 $y = 2e^{2x} + e^{-x}$

Curve cuts the y-axis at $x = 0$:

$y = 2e^0 + e^0 = 2 + 1 = 3$

The coordinates of P are $(0, 3)$.

$$\frac{dy}{dx} = 2e^{2x}(2) + e^{-x}(-1)$$
$$= 4e^{2x} - e^{-x}$$

At P:

$$\frac{dy}{dx} = 4e^0 - e^0 = 4 - 1 = 3$$

Equation of the tangent is:

$$y - 3 = 3(x - 0)$$
$$y = 3x + 3$$

The tangent crosses the x-axis when $y = 0$:

$3x + 3 = 0$

$3x = -3$

$x = -1$

The tangent crosses the x-axis at the point $(-1, 0)$.

8 $y = (x - 4)e^x$

$$\frac{dy}{dx} = (x - 4)e^x + e^x(1)$$
$$= (x - 4 + 1)e^x$$
$$= (x - 3)e^x$$

Stationary point when $\frac{dy}{dx} = 0$:

$(x - 3)e^x = 0$

$e^x > 0$, so $x - 3 = 0$

$x = 3$

$y = (3 - 4)e^3 = -e^3$

Stationary point has coordinates $(3, -e^3)$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[(x - 3)e^x]$$
$$= (x - 3)e^x + e^x(1)$$
$$= (x - 3 + 1)e^x$$
$$= (x - 2)e^x$$

At $x = 3$:

$$\frac{d^2y}{dx^2} = (3 - 2)e^3 = e^3 > 0$$

So this is a minimum point.

Remember that the second derivative shows the rate of change of gradient. If, at a minimum point, the second derivative is positive then the gradient must be increasing. In other words, it has changed from negative to positive at the stationary point, so it is a minimum point.

9 $$y = \frac{e^{2x}}{x^2}$$

$$\frac{dy}{dx} = \frac{x^2e^{2x}(2) - e^{2x}(2x)}{x^4}$$
$$= \frac{2x^2e^{2x} - 2xe^{2x}}{x^4}$$

Stationary when $\frac{dy}{dx} = 0$:

$$\frac{2x^2e^{2x} - 2xe^{2x}}{x^4} = 0$$
$$2x^2e^{2x} - 2xe^{2x} = 0$$
$$2xe^{2x}(x-1) = 0$$

$x = 0$ or $x - 1 = 0$

But $x \neq 0$ because the equation of the curve is not defined for $x = 0$.

So $x = 1$

$$y = \frac{e^2}{1} = e^2$$

Stationary point is at $(1, e^2)$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{2x^2e^{2x} - 2xe^{2x}}{x^4}\right]$$
$$= \frac{d}{dx}\left[(2x^{-2} - 2x^{-3})e^{2x}\right]$$
$$= (2x^{-2} - 2x^{-3})e^{2x}(2) + e^{2x}(-4x^{-3} + 6x^{-4})$$
$$= 4e^{2x}(x^{-2} - x^{-3}) + 2e^{2x}(-2x^{-3} + 3x^{-4})$$

When $x = 1$:

$$\frac{d^2y}{dx^2} = 4e^2(0) + 2e^2(-2+3) = 2e^2 > 0$$

So this is a minimum point.

10 **a** $$y = x^2e^{-x}$$

$$\frac{dy}{dx} = x^2e^{-x}(-1) + e^{-x}(2x)$$
$$= xe^{-x}(2-x)$$

Stationary points when $\frac{dy}{dx} = 0$:

$x = 0$ or $x = 2$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[xe^{-x}(2-x)]$$
$$= xe^{-x}(-1) + x(2-x)e^{-x}(-1) + e^{-x}(2-x)$$
$$= e^{-x}[-x - x(2-x) + 2 - x]$$
$$= e^{-x}(x^2 - 4x + 2)$$

When $x = 0$:

$$\frac{d^2y}{dx^2} = e^0(0 - 0 + 2) = 2 > 0$$

So there is a minimum point at $x = 0$.

When $x = 2$:

$$\frac{d^2y}{dx^2} = e^{-2}(2^2 - 4(2) + 2) = -2e^{-2} < 0$$

So there is a maximum point at $x = 2$.

b Using the expression for $\frac{dy}{dx}$ from part **a**, the gradient of the tangent at $x = 1$ is:

$$1e^{-1}(2-1) = e^{-1} = \frac{1}{e}$$

Gradient of the normal $= -\frac{1}{\left(\frac{1}{e}\right)} = -e$

$x = 1$ gives $y = 1\mathrm{e}^{-1} = \mathrm{e}^{-1} = \dfrac{1}{\mathrm{e}}$

Equation of the normal at $\left(1, \dfrac{1}{\mathrm{e}}\right)$ is:

$$y - \frac{1}{\mathrm{e}} = -\mathrm{e}(x - 1)$$
$$\mathrm{e}y - 1 = -\mathrm{e}^2 x + \mathrm{e}^2$$
$$\mathrm{e}^2 x + \mathrm{e}y = 1 + \mathrm{e}^2$$

> It can be tempting to use decimals instead of 'e' in questions like these. You must resist, because decimals will only give approximate answers.

11 $y = x^2\mathrm{e}^{-2x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2\mathrm{e}^{-2x}(-2) + \mathrm{e}^{-2x}(2x)$$
$$= 2x\mathrm{e}^{-2x}(1 - x)$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2x\mathrm{e}^{-2x}(-1) + 2x(1 - x)\mathrm{e}^{-2x}(-2) + \mathrm{e}^{-2x}(1 - x)(2)$$
$$= 2\mathrm{e}^{-2x}[-x - 2x(1 - x) + 1 - x]$$
$$= 2\mathrm{e}^{-2x}(2x^2 - 4x + 1)$$

When $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$:

$$2x^2 - 4x + 1 = 0$$
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(1)}}{2(2)}$$
$$= \frac{4 \pm \sqrt{8}}{4}$$
$$= 1 \pm \frac{\sqrt{4}\sqrt{2}}{4}$$
$$= 1 \pm \frac{\sqrt{2}}{2}$$
$$= 1 \pm \frac{1}{\sqrt{2}}$$

12 $y = \dfrac{\mathrm{e}^{2x-1}}{x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x\mathrm{e}^{2x-1}(2) - \mathrm{e}^{2x-1}(1)}{x^2}$$
$$= \frac{2x\mathrm{e}^{2x-1} - \mathrm{e}^{2x-1}}{x^2}$$

When $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$\frac{2x\mathrm{e}^{2x-1} - \mathrm{e}^{2x-1}}{x^2} = 0$$
$$2x\mathrm{e}^{2x-1} - \mathrm{e}^{2x-1} = 0$$
$$\mathrm{e}^{2x-1}(2x - 1) = 0$$

$2x - 1 = 0$ because $\mathrm{e}^{2x-1} \neq 0$.

$$x = \frac{1}{2}$$
$$y = \frac{\mathrm{e}^{2x-1}}{x} = \frac{\mathrm{e}^{2\left(\frac{1}{2}\right)-1}}{\left(\frac{1}{2}\right)} = 2\mathrm{e}^0 = 2$$

Stationary point has coordinates $\left(\dfrac{1}{2}, 2\right)$.

E **13** $\dfrac{\mathrm{d}}{\mathrm{d}x}(2^x)$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left[(\mathrm{e}^{\ln 2})^{x}\right]$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x \ln 2})$$
$$= \mathrm{e}^{x \ln 2}(\ln 2)$$
$$= 2^{x} \ln 2$$

14 $\frac{\mathrm{d}}{\mathrm{d}x}(3^{x})$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left[(\mathrm{e}^{\ln 3})^{x}\right]$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x \ln 3})$$
$$= \mathrm{e}^{x \ln 3}(\ln 3)$$
$$= 3^{x} \ln 3$$

$$y = x(3^{x})$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(3^{x} \ln 3) + 3^{x}(1)$$
$$= 3^{x}(x \ln 3 + 1)$$

When $x = 1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3^{1}(1 \ln 3 + 1)$$
$$= 3 \ln 3 + 3$$

EXERCISE 4D

1 **d** $y = 5 + \ln(x^2 + 1)$

$$\frac{dy}{dx} = 0 + \frac{1}{x^2+1}(2x)$$
$$= \frac{2x}{1+x^2}$$

j $y = \ln(\ln x)$

$$\frac{dy}{dx} = \frac{1}{\ln x}\left(\frac{1}{x}\right)$$
$$= \frac{1}{x \ln x}$$

2 **Method 1**

For part **a**:

$$y = \ln 3x = \ln 3 + \ln x$$
$$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

For part **b**:

$$y = \ln 7x = \ln 7 + \ln x$$
$$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

Given that $\ln 3$ and $\ln 7$ are both constants, they differentiate to give zero. So differentiating either function is the same as just differentiating the $\ln x$ that both have in common.

Method 2

This is really the same as method 1, but shows a slightly more general case.

$$y = \ln kx = \ln k + \ln x$$
$$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

because $\ln k$ is a constant.

It does not matter what the value of k is; the answer will be the same.

3 **b** $y = 2x^3 \ln x$

$$\frac{dy}{dx} = 2x^3\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}(2x^3)$$
$$= 2x^3\left(\frac{1}{x}\right) + \ln x(6x^2)$$
$$= 2x^2 + 6x^2 \ln x$$
$$= 2x^2(1 + 3 \ln x)$$

f $y = \dfrac{\ln 5x}{x}$

$$\frac{dy}{dx} = \frac{x\frac{d}{dx}[\ln 5x] - \ln 5x\frac{d}{dx}[x]}{x^2}$$
$$= \frac{x\left[\frac{1}{5x}(5)\right] - \ln 5x}{x^2}$$
$$= \frac{1 - \ln 5x}{x^2}$$

4

When a question asks you to sketch a curve, you should always try to find important points on the graph, such as intersections with axes or stationary points.

a $y = \ln(2x - 3)$

The curve crosses the x-axis when $y = 0$:

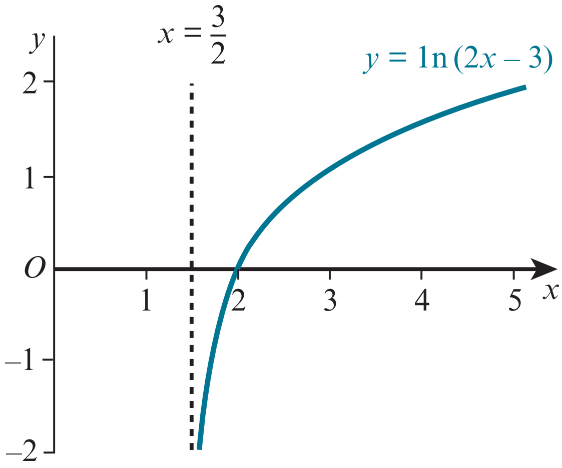
$$\ln(2x-3)=0$$
$$2x-3=1$$
$$2x=4$$
$$x=2$$

The curve crosses the y-axis when $x=0$:

$y=\ln(0-3)=\ln(-3)$

It is not possible to take the logarithm of a negative number, so $x>\frac{3}{2}$.

The curve does not cross the y-axis.



b $y=\ln(2x-3)$

$$\frac{dy}{dx}=\frac{1}{2x-3}(2)=\frac{2}{2x-3}$$

When $x=5$:

$$\frac{dy}{dx}=\frac{2}{2(5)-3}=\frac{2}{7}$$

5 $y=e^{2x}-5\ln(2x+1)$

$$\frac{dy}{dx}=e^{2x}(2)-\frac{5}{2x+1}(2)$$
$$=2e^{2x}-\frac{10}{2x+1}$$

At $x=0$:

$$\frac{dy}{dx}=2e^{0}-\frac{10}{1}=2-10=-8$$

6 $y=x^2\ln 5x$

$$\frac{dy}{dx}=x^2\left(\frac{1}{x}\right)+2x\ln 5x$$
$$=x+2x\ln 5x$$
$$\frac{d^2y}{dx^2}=\frac{d}{dx}(x+2x\ln 5x)$$
$$=1+2x\left(\frac{1}{x}\right)+2\ln 5x$$
$$=3+2\ln 5x$$

At $x=2$:

$$\frac{dy}{dx}=2+2(2)\ln 10=2+4\ln 10$$
$$\frac{d^2y}{dx^2}=3+2\ln 10$$

7 $y=x^2\ln x$

$$\frac{dy}{dx}=x^2\left(\frac{1}{x}\right)+2x\ln x$$
$$=x+2x\ln x$$

There is a stationary point where $\frac{dy}{dx} = 0$:

$$x + 2x\ln x = 0$$
$$x(1 + 2\ln x) = 0$$
$$\ln x = -\frac{1}{2} \text{ or } x = 0$$
$$x = e^{-\frac{1}{2}}$$

Not valid as you cannot take the log of zero.

When $x = e^{-\frac{1}{2}}$:

$$y = \left(e^{-\frac{1}{2}}\right)^2 \ln\left(e^{-\frac{1}{2}}\right)$$
$$= -\frac{1}{2}e^{-1} = -\frac{1}{2e}$$

Stationary point is:

$$\left(e^{-\frac{1}{2}}, -\frac{1}{2}e^{-1}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[x + 2x\ln x]$$
$$= 1 + 2x\left(\frac{1}{x}\right) + 2\ln x$$
$$= 3 + 2\ln x$$

When $x = e^{-\frac{1}{2}}$:

$$\frac{d^2y}{dx^2} = 3 + 2\ln\left(e^{-\frac{1}{2}}\right) = 3 + 2\left(-\frac{1}{2}\right) = 2 > 0$$

So this is a minimum point.

8 $$y = \frac{\ln x}{x}$$

$$\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2}$$
$$= \frac{1 - \ln x}{x^2}$$

Stationary when $\frac{dy}{dx} = 0$:

$$\frac{1 - \ln x}{x^2} = 0$$
$$1 - \ln x = 0$$
$$\ln x = 1$$
$$x = e^1 = e$$

When $x = e$:

$$y = \frac{\ln e}{e} = \frac{1}{e}$$

There is a stationary point at $\left(e, \frac{1}{e}\right)$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1 - \ln x}{x^2}\right]$$
$$= \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)(2x)}{x^4}$$
$$= \frac{-x - 2x(1 - \ln x)}{x^4}$$

When $x = e$:

$$\frac{d^2y}{dx^2} = \frac{-e - 2e(1 - \ln e)}{e^4}$$
$$= \frac{-e - 2e + 2e}{e^4} = -\frac{1}{e^3} < 0$$

So this is a maximum point.

If you need to differentiate twice, take the time to tidy up as you go. It is much easier to differentiate a fully simplified expression.

9

$$y = \ln(5x - 4)$$
$$\frac{dy}{dx} = \frac{1}{5x - 4}(5)$$
$$= \frac{5}{5x - 4}$$

When $x = 1$:

$y = \ln(5 - 4) = \ln 1 = 0$

The point is $(1, 0)$

$$\frac{dy}{dx} = \frac{5}{5(1) - 4} = 5$$

Equation of the tangent is:

$y - 0 = 5(x - 1)$

$y = 5x - 5$

10 **b**

$$y = \ln\left(\frac{1}{3x + 2}\right)$$
$$= \ln(3x + 2)^{-1}$$
$$= -\ln(3x + 2)$$
$$\frac{dy}{dx} = -\frac{1}{3x + 2}(3)$$
$$= -\frac{3}{3x + 2}$$

g

$$y = \ln\left[\frac{3 - x}{(x + 4)(x - 1)}\right]$$
$$= \ln(3 - x) - \ln(x + 4) - \ln(x - 1)$$
$$\frac{dy}{dx} = \frac{-1}{3 - x} - \frac{1}{x + 4} - \frac{1}{x - 1}$$
$$= \frac{1}{x - 3} - \frac{1}{x + 4} - \frac{1}{x - 1}$$

11 **a**

$$e^y = 2x^2 - 1$$
$$y = \ln(2x^2 - 1)$$
$$\frac{dy}{dx} = \frac{1}{2x^2 - 1}(4x)$$
$$= \frac{4x}{2x^2 - 1}$$

b

$$e^y = 3x^3 + 2x$$
$$y = \ln(3x^3 + 2x)$$
$$\frac{dy}{dx} = \frac{1}{3x^3 + 2x}(9x^2 + 2)$$
$$= \frac{9x^2 + 2}{3x^3 + 2x}$$

c
$$\begin{aligned} e^y &= (x+1)(x-5) \\ y &= \ln[(x+1)(x-5)] \\ &= \ln(x+1) + \ln(x-5) \\ \frac{dy}{dx} &= \frac{1}{x+1} + \frac{1}{x-5} \\ &= \frac{x-5+x+1}{(x+1)(x-5)} \\ &= \frac{2x-4}{(x+1)(x-5)} \end{aligned}$$

12
$$\begin{aligned} x &= \frac{1}{5}\left[e^{y(2x-3)} + 4\right] \\ 5x &= e^{y(2x-3)} + 4 \\ e^{y(2x-3)} &= 5x - 4 \\ y(2x-3) &= \ln(5x-4) \\ y &= \frac{\ln(5x-4)}{2x-3} \\ \frac{dy}{dx} &= \frac{(2x-3)\left(\frac{5}{5x-4}\right) - 2\ln(5x-4)}{(2x-3)^2} \\ &= \frac{5(2x-3) - 2(5x-4)\ln(5x-4)}{(2x-3)^2(5x-4)} \end{aligned}$$

When $x = 1$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{5(-1) - 2(1)\ln(1)}{(-1)^2(1)} \\ &= -\frac{5}{1} = -5 \end{aligned}$$

EXERCISE 4E

1 **e** $y = 4\tan 5x$

$$\frac{dy}{dx} = 4\sec^2(5x)(5)$$
$$= 20\sec^2(5x)$$

h $y = \sin\left(2x + \frac{\pi}{3}\right)$

$$\frac{dy}{dx} = \cos\left(2x + \frac{\pi}{3}\right)(2)$$
$$= 2\cos\left(2x + \frac{\pi}{3}\right)$$

2

Notation like $\sin^2 x$ or $\sec^2 x$ is a nice shorthand but it does hide what you are differentiating a little. If you are ever unsure, write out what the shorthand means. worked solution **2b** shows you how this might be done.

b $y = 5\cos^2 3x = 5(\cos 3x)^2$

$$\frac{dy}{dx} = 10(\cos 3x)^1(-\sin 3x)(3)$$
$$= -30\sin 3x\cos 3x$$
$$= -15(2\sin 3x\cos 3x)$$
$$= -15\sin 6x$$

d $y = (3 - \cos x)^4$

$$\frac{dy}{dx} = 4(3 - \cos x)^3(- - \sin x)$$
$$= 4\sin x(3 - \cos x)^3$$

3 **f** $y = \frac{x}{\cos x}$

Using the quotient rule:

$$\frac{dy}{dx} = \frac{\cos x - x(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos x + x\sin x}{\cos^2 x}$$
$$= \frac{1}{\cos x}\left[\frac{\cos x}{\cos x} + \frac{x\sin x}{\cos x}\right]$$
$$= \sec x(1 + x\tan x)$$

j

In worked solution **3 j**, the fact that $\frac{d}{dx}(\operatorname{cosec} x) = -\cot x \operatorname{cosec} x$ is used. See worked solution **9 b** for a proof of this result.

$$y = \frac{1}{\sin^3 2x}$$
$$= \operatorname{cosec}^3 2x$$
$$= (\operatorname{cosec} 2x)^3$$
$$\frac{dy}{dx} = 3(\operatorname{cosec}^2 2x)(-\cot 2x \operatorname{cosec} 2x)(2)$$
$$= -6\cot 2x \operatorname{cosec}^3 2x$$

4 **c** $y = e^{\tan 3x}$

$$\frac{dy}{dx} = e^{\tan 3x}(\sec^2 3x)(3)$$
$$= 3\sec^2(3x)e^{\tan 3x}$$

j $y = x \ln(\sin x)$

$$\frac{dy}{dx} = x\left(\frac{1}{\sin x}(\cos x)\right) + \ln(\sin x)$$

$$= x \cot x + \ln(\sin x)$$

5 $y = 3\sin 2x - 5\tan x$

$$\frac{dy}{dx} = 3\cos 2x(2) - 5\sec^2 x$$

$$= 6\cos 2x - 5\sec^2 x$$

when $x = 0$:

$$\frac{dy}{dx} = 6\cos 0 - \frac{5}{\cos^2 0}$$

$$= 6 - 5 = 1$$

6 $y = 2\sin 3x - 4\cos x$

$$\frac{dy}{dx} = 2\cos 3x(3) - 4(-\sin x)$$

$$= 6\cos 3x + 4\sin x$$

When $x = \frac{\pi}{3}$:

$$\frac{dy}{dx} = 6\cos\pi + 4\sin\left(\frac{\pi}{3}\right)$$

$$= -6 + 4\frac{\sqrt{3}}{2} = -6 + 2\sqrt{3}$$

7 $y = \sin^2 x = (\sin x)^2$

$$\frac{dy}{dx} = 2\sin x(\cos x) = \sin 2x$$

When the gradient $= \frac{\sqrt{3}}{2}$:

$$\sin 2x = \frac{\sqrt{3}}{2}$$

$2x = \frac{\pi}{3}$ or $2x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$x = \frac{\pi}{6}$ or $x = \frac{\pi}{3}$

8 $y = \frac{5}{2 - \tan x} = 5(2 - \tan x)^{-1}$

$$\frac{dy}{dx} = -5(2 - \tan x)^{-2}(-\sec^2 x)$$

$$= \frac{5\sec^2 x}{(2 - \tan x)^2}$$

Given that squared numbers are never negative:

$$\frac{dy}{dx} = \frac{\text{positive}}{\text{positive}} = \text{positive}$$

In worked solution **8**, you should note that the gradient is always positive *when it is defined.* When $\tan x = 2$ there is a division by zero, which is not allowed. So, the gradient is positive as long as $\tan x$ is not 2.

9 **a** $\frac{d}{dx}(\sec x)$

$$= \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$
$$= \frac{d}{dx}(\cos x)^{-1}$$
$$= -(\cos x)^{-2}(-\sin x)$$
$$= \frac{\sin x}{\cos^2 x}$$
$$= \frac{1}{\cos x}\frac{\sin x}{\cos x}$$
$$= \sec x \tan x$$

b $\frac{d}{dx}(\operatorname{cosec} x)$

$$= \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$
$$= \frac{d}{dx}(\sin x)^{-1}$$
$$= -(\sin x)^{-2}(\cos x)$$
$$= -\frac{\cos x}{\sin^2 x}$$
$$= -\frac{1}{\sin x}\frac{\cos x}{\sin x}$$
$$= -\cot x \operatorname{cosec} x$$

c $\frac{d}{dx}(\cot x)$

$$= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$
$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$
$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$
$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}$$
$$= -\frac{1}{\sin^2 x}$$
$$= -\operatorname{cosec}^2 x$$

Identities are always very helpful when differentiating trigonometric functions. If you find that your answer looks different to that given in the back of your coursebook, try to use identities to rewrite your answer so that it looks the same. If you can memorise the identities, then you are more likely to spot when to use them.

10 $y = x \sin x$

$$\frac{dy}{dx} = x\cos x + \sin x$$

When $x = \frac{\pi}{2}$:

$$\frac{dy}{dx} = \left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)$$
$$= 0 + 1 = 1$$

Gradient of the normal $= -\frac{1}{1} = -1$

Equation of the normal is:

$$y - \frac{\pi}{2} = -1\left(x - \frac{\pi}{2}\right)$$
$$y - \frac{\pi}{2} = -x + \frac{\pi}{2}$$
$$y = -x + \pi$$

The normal intersects the x-axis when $y = 0$:

$$-x + \pi = 0$$
$$x = \pi$$

Intersection is at $(\pi, 0)$.

11 $y = 5\sin 3x - 2\cos x$

$\frac{dy}{dx} = 15\cos 3x + 2\sin x$

When $x = \frac{\pi}{3}$:

$$\frac{dy}{dx} = 15\cos\pi + 2\sin\left(\frac{\pi}{3}\right)$$
$$= -15 + 2\left(\frac{\sqrt{3}}{2}\right) = -15 + \sqrt{3}$$

Equation of the tangent is:

$$y - (-1) = (-15 + \sqrt{3})\left(x - \frac{\pi}{3}\right)$$

$y = -13.3x + 12.9$ (to 3 significant figures)

12 $y = 3\cos 2x + 4\sin 2x + 1$

$\frac{dy}{dx} = -6\sin 2x + 8\cos 2x$

Stationary points when $\frac{dy}{dx} = 0$:

$$-6\sin 2x + 8\cos 2x = 0$$
$$6\sin 2x = 8\cos 2x$$
$$\tan 2x = \frac{8}{6} = \frac{4}{3}$$

$2x = 0.927\ldots$ or $2x = 4.068\ldots$

$x = 0.464$ (to 3 s.f.) $x = 2.03$ (to 3 s.f.)

> Remember to check which mode you have left your calculator in. Only radians can be used when differentiating trigonometric functions.

13 $y = e^x\cos x$

$$\frac{dy}{dx} = e^x(-\sin x) + e^x\cos x$$
$$= e^x(\cos x - \sin x)$$

Stationary point when $\frac{dy}{dx} = 0$:

$e^x(\cos x - \sin x) = 0$

$e^x > 0$, so $\cos x - \sin x = 0$

$\cos x = \sin x$

$\tan x = 1$

$x = \frac{\pi}{4}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[e^x(\cos x - \sin x)]$$
$$= e^x(-\sin x - \cos x) + e^x(\cos x - \sin x)$$
$$= e^x(-2\sin x)$$
$$= -2e^x\sin x$$

When $x = \frac{\pi}{4}$:

$$\frac{d^2y}{dx^2} = -2e^{\frac{\pi}{4}}\sin\left(\frac{\pi}{4}\right)$$
$$= -\sqrt{2}e^{\frac{\pi}{4}} < 0$$

So this is a maximum point.

14 $y = \frac{\sin 2x}{e^{2x}}$

$$\frac{dy}{dx} = \frac{2e^{2x}\cos 2x - 2e^{2x}\sin 2x}{e^{4x}}$$

Stationary point when $\frac{dy}{dx} = 0$:

$$\frac{2e^{2x}\cos 2x - 2e^{2x}\sin 2x}{e^{4x}} = 0$$

$$2e^{2x}\cos 2x - 2e^{2x}\sin 2x = 0$$

$$2e^{2x}(\cos 2x - \sin 2x) = 0$$

$$\cos 2x - \sin 2x = 0$$

$$\cos 2x = \sin 2x$$

$$\tan 2x = 1$$

$$2x = \frac{\pi}{4}$$

$$x = \frac{\pi}{8}$$

15 $y = \frac{e^{3x}}{\sin 3x}$

$$\frac{dy}{dx} = \frac{3e^{3x}\sin 3x - 3e^{3x}\cos 3x}{\sin^2 3x}$$

Stationary point when $\frac{dy}{dx} = 0$:

$$\frac{3e^{3x}\sin 3x - 3e^{3x}\cos 3x}{\sin^2 3x} = 0$$

$$3e^{3x}\sin 3x - 3e^{3x}\cos 3x = 0$$

$$3e^{3x}(\sin 3x - \cos 3x) = 0$$

$$\sin 3x = \cos 3x$$

$$\tan 3x = 1$$

$$3x = \frac{\pi}{4}$$

$$x = \frac{\pi}{12}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{3e^{3x}\sin 3x - 3e^{3x}\cos 3x}{\sin^2 3x}\right]$$

You could certainly work out this second derivative, but it would take a very long time. Instead, you can try values a very small distance on either side of the turning point as shown here.

When $x = \frac{\pi}{12}$:

$$y = \frac{e^{\frac{\pi}{4}}}{\sin\frac{\pi}{4}} = 3.1017\ldots$$

When $x = \frac{\pi}{12} + 0.1$:

$$3x = \frac{\pi}{4} + 0.3$$

$$y = \frac{e^{\frac{\pi}{4}+0.3}}{\sin\left(\frac{\pi}{4} + 0.3\right)} = 3.3472\ldots > 3.1017\ldots$$

The stationary point is a minimum.

16 $y = \sin 2x - x$

$$\frac{dy}{dx} = 2\cos 2x - 1$$

$$\frac{d^2y}{dx^2} = -4\sin 2x$$

Stationary points when $\frac{dy}{dx} = 0$:

$$2\cos 2x - 1 = 0$$

$$\cos 2x = \frac{1}{2}$$

$$2x = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{6}$$

When $x = \frac{\pi}{6}$: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4\sin\frac{\pi}{3} < 0$, so this is a maximum point

or $2x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \Rightarrow x = \frac{5\pi}{6}$

When $x = \frac{5\pi}{6}$: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4\sin\frac{5\pi}{3} > 0$, so this is a minimum point

or $2x = \frac{\pi}{3} + 2\pi = \frac{7\pi}{3} \Rightarrow x = \frac{7\pi}{6}$

When $x = \frac{7\pi}{6}$: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4\sin\frac{7\pi}{3} < 0$, so this is a maximum point

or $2x = \frac{5\pi}{3} + 2\pi = \frac{11\pi}{3} \Rightarrow x = \frac{11\pi}{6}$

When $x = \frac{11\pi}{6}$: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4\sin\frac{11\pi}{3} > 0$, so this is a minimum point.

17 $y = \tan x \cos 2x$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \tan x(-2\sin 2x) + \sec^2 x\cos 2x \\ &= -2\tan x\sin 2x + \sec^2 x\cos 2x\end{aligned}$$

Stationary point when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$\begin{aligned}-2\tan x\sin 2x + \sec^2 x\cos 2x &= 0 \\ -2\frac{\sin x}{\cos x}(2\sin x\cos x) + \frac{\cos 2x}{\cos^2 x} &= 0 \\ -4\sin^2 x + \frac{1-2\sin^2 x}{1-\sin^2 x} &= 0 \\ -4\sin^2 x(1-\sin^2 x) + 1 - 2\sin^2 x &= 0 \\ 4(\sin^2 x)^2 - 6\sin^2 x + 1 &= 0\end{aligned}$$

Letting $u = \sin^2 x$:

$4u^2 - 6u + 1 = 0$

$$\begin{aligned}u &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(4)(1)}}{2(4)} \\ &= \frac{6 \pm \sqrt{20}}{8} \\ &= \frac{6 \pm 2\sqrt{5}}{8} \\ &= \frac{3 \pm \sqrt{5}}{4}\end{aligned}$$

$\sin^2 x = \frac{3 \pm \sqrt{5}}{4}$

But $\frac{3+\sqrt{5}}{4} > 1$

$\sin^2 x = \frac{3-\sqrt{5}}{4}$

$\sin x = \pm\sqrt{\frac{3-\sqrt{5}}{4}}$

But

$0 \leqslant x < \frac{\pi}{2}$

$\sin x \geqslant 0$

$\sin x = \sqrt{\frac{3-\sqrt{5}}{4}}$

$x = 0.452$ (to 3 significant figures)

EXERCISE 4F

1 b $\frac{\mathrm{d}}{\mathrm{d}x}(x^3 + 2y^2)$

$= 3x^2 + 4y\frac{\mathrm{d}y}{\mathrm{d}x}$

k $\frac{\mathrm{d}}{\mathrm{d}x}(5y + \mathrm{e}^x \sin y)$

$= 5\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^x \cos y\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^x \sin y$

2 d

$$x \ln y = 2x + 5$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x \ln y) = \frac{\mathrm{d}}{\mathrm{d}x}(2x + 5)$$

$$x\left(\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \ln y = 2$$

$$\frac{x}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \ln y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(2 - \ln y)}{x}$$

In the worked solutions for Question **2** it is shown that you take $\frac{\mathrm{d}}{\mathrm{d}x}$ of both sides in each case. It is, however, obvious that this has happened from the next line. This step has been included to make it easier for you to follow what is going on, but you don't need to write it yourself.

g

$$xy^3 = 2 \ln y$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy^3) = \frac{\mathrm{d}}{\mathrm{d}x}(2 \ln y)$$

$$x\left(3y^2\frac{\mathrm{d}y}{\mathrm{d}x}\right) + y^3 = \frac{2}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{2}{y}\frac{\mathrm{d}y}{\mathrm{d}x} - 3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} = y^3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{2}{y} - 3xy^2\right) = y^3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{2 - 3xy^3}{y}\right) = y^3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^4}{2 - 3xy^3}$$

3

$$x^2 + 3xy - 5y + y^3 = 22$$

$$2x + 3x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(3x - 5 + 3y^2) = -2x - 3y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x - 3y}{3x - 5 + 3y^2}$$

At (1, 3), when $x = 1$ and $y = 3$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2 - 9}{3 - 5 + 27}$$

$$= -\frac{11}{25}$$

Before you studied implicit differentiation, your expressions for $\frac{\mathrm{d}y}{\mathrm{d}x}$ usually contained only one variable. Implicit differentiation usually leads to expressions for $\frac{\mathrm{d}y}{\mathrm{d}x}$ that contain both x and y terms. This means that you need to know BOTH the x- and y-coordinates at any point where you want to find the gradient.

4

$$2x^3 - 4xy + y^3 = 16$$
$$6x^2 - 4x\frac{dy}{dx} - 4y + 3y^2\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(4x - 3y^2) = 6x^2 - 4y$$
$$\frac{dy}{dx} = \frac{6x^2 - 4y}{4x - 3y^2}$$

The curve crosses the x-axis when $y = 0$:

$$2x^3 = 16$$
$$x^3 = 8$$
$$x = 2$$

When $x = 2$ and $y = 0$:

$$\frac{dy}{dx} = \frac{6(2^2) - 4(0)}{4(2) - 3(0)}$$
$$= \frac{24}{8} = 3$$

5

$$2x^2 + 3y^2 - 2x + 4y = 4$$
$$4x + 6y\frac{dy}{dx} - 2 + 4\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(6y + 4) = 2 - 4x$$
$$\frac{dy}{dx} = \frac{2 - 4x}{6y + 4}$$

At the point $(1, -2)$:

$$\frac{dy}{dx} = \frac{2 - 4(1)}{6(-2) + 4}$$
$$= \frac{-2}{-12 + 4}$$
$$= \frac{1}{4}$$

Tangent at the point $(1, -2)$ is:

$$y - (-2) = \frac{1}{4}(x - 1)$$
$$y + 2 = \frac{1}{4}x - \frac{1}{4}$$
$$y = \frac{1}{4}x - \frac{9}{4}$$

6

$$4x^2y + 8\ln x + 2\ln y = 4$$
$$4x^2\frac{dy}{dx} + 8xy + \frac{8}{x} + \frac{2}{y}\frac{dy}{dx} = 0$$
$$\left(4x^2 + \frac{2}{y}\right)\frac{dy}{dx} = -8xy - \frac{8}{x}$$
$$\left(\frac{4x^2y + 2}{y}\right)\frac{dy}{dx} = \frac{-8x^2y - 8}{x}$$
$$\frac{dy}{dx} = \left(\frac{-8x^2y - 8}{x}\right)\left(\frac{y}{4x^2y + 2}\right)$$
$$= -\frac{8}{2}\left(\frac{x^2y + 1}{x}\right)\left(\frac{y}{2x^2y + 1}\right)$$
$$= -\frac{4y(x^2y + 1)}{x(2x^2y + 1)}$$

At the point $(1,\ 1)$ when $x = 1$ and $y = 1$:

$$\frac{dy}{dx} = -\frac{4(1 + 1)}{1(2 + 1)} = -\frac{8}{3}$$

Gradient of the normal $= -\dfrac{1}{\left(-\frac{8}{3}\right)} = \dfrac{3}{8}$

Equation of the normal is:

$$y - 1 = \frac{3}{8}(x - 1)$$
$$y = \frac{3}{8}x - \frac{3}{8} + \frac{8}{8}$$
$$y = \frac{3}{8}x + \frac{5}{8}$$

7 **a**

$$5x^2 + 2xy + 2y^2 = 45$$
$$10x + 2x\frac{dy}{dx} + 2y + 4y\frac{dy}{dx} = 0$$
$$(2x + 4y)\frac{dy}{dx} = -2y - 10x$$
$$\frac{dy}{dx} = -\frac{(2y + 10x)}{(2x + 4y)}$$
$$= -\frac{y + 5x}{x + 2y}$$

Tangent is parallel to x-axis when the gradient $= 0$:

$$-\frac{y + 5x}{x + 2y} = 0$$
$$y + 5x = 0$$
$$y = -5x$$

b

The points lie on the line with equation $y = -5x$, but also on the curve. You need to look for the points at which this line and the curve intersect.

Substituting the equation of the line into the equation of the curve:

$$5x^2 + 2x(-5x) + 2(-5x)^2 = 45$$
$$5x^2 - 10x^2 + 50x^2 = 45$$
$$45x^2 = 45$$
$$x^2 = 1$$

$x = 1$ or $x = -1$

$y = -5x = -5$ $\quad$ $y = -5x = 5$

The points are $(1, -5)$ and $(-1, 5)$.

8 **a** When $x = 4$:

$$y^2 - 4xy - x^2 = 20$$
$$y^2 - 16y - 16 = 20$$
$$y^2 - 16y - 36 = 0$$
$$(y - 18)(y + 2) = 0$$

$y = 18$ or $y = -2$

The points are $(4, 18)$ and $(4, -2)$.

b

$$y^2 - 4xy - x^2 = 20$$
$$2y\frac{dy}{dx} - 4x\frac{dy}{dx} - 4y - 2x = 0$$
$$(2y - 4x)\frac{dy}{dx} = 4y + 2x$$
$$\frac{dy}{dx} = \frac{4y + 2x}{2y - 4x} = \frac{2y + x}{y - 2x}$$

At the point $(4, -2)$:

$2y + x = -4 + 4 = 0$

The curve is parallel to the x-axis at this point.

c At the point $(4, 18)$

$$\frac{dy}{dx} = \frac{36 + 4}{18 - 8} = \frac{40}{10} = 4$$

Equation of the tangent is:

$$y - 18 = 4(x - 4)$$
$$y = 4x - 16 + 18$$
$$y = 4x + 2$$

9 **a**
$$y^3 - 12xy + 16 = 0$$
$$3y^2\frac{dy}{dx} - 12x\frac{dy}{dx} - 12y = 0$$
$$(3y^2 - 12x)\frac{dy}{dx} = 12y$$
$$\frac{dy}{dx} = \frac{12y}{3y^2 - 12x} = \frac{4y}{y^2 - 4x}$$

If the curve had stationary points then $\frac{4y}{y^2 - 4x} = 0$ at such a point.

$y = 0$; but no such point can lie on the curve with equation $y^3 - 12xy + 16 = 0$ because if $y = 0$ then this gives $16 = 0$, which is not possible.

There are no stationary points on this curve.

b

If the denominator of $\frac{dy}{dx}$ is not zero, then there is finite gradient and the line cannot be vertical. If, however, the denominator is zero, the gradient cannot be finite, so the line is vertical.

Parallel to the y-axis when the denominator of $\frac{dy}{dx}$ is zero.

When the denominator of $\frac{4y}{y^2 - 4x}$ is zero:

$$y^2 - 4x = 0$$
$$y^2 = 4x$$

The curve has equation

$$y^3 - 12xy + 16 = 0$$
$$y(y^2 - 12x) = -16$$
$$y = -\frac{16}{y^2 - 12x} = \frac{16}{4x - 12x} = -\frac{2}{x}$$
$$y^2 = \frac{4}{x^2}$$

Equating the two expressions for y^2:

$$4x = \frac{4}{x^2}$$
$$x^3 = 1$$
$$x = 1$$
$$y = \sqrt{4x} = \sqrt{4} = 2$$

The point has coordinates (1, 2).

10
$$5e^x y^2 + 2e^x y = 88$$
$$5y^2e^x + 10e^x y\frac{dy}{dx} + 2e^x y + 2e^x\frac{dy}{dx} = 0$$
$$2e^x(5y + 1)\frac{dy}{dx} = -ye^x(2 + 5y)$$
$$\frac{dy}{dx} = -\frac{y(2 + 5y)}{2(5y + 1)}$$

At the point (0, 4) when $x = 0$ and $y = 4$:

$$\frac{dy}{dx} = -\frac{4(2 + 20)}{2(20 + 1)} = -\frac{4(22)}{2(21)} = -\frac{44}{21}$$

11

$$x^2 - 4x + 6y + 2y^2 = 12$$

$$2x - 4 + 6\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(6 + 4y) = 4 - 2x$$

$$\frac{dy}{dx} = \frac{4 - 2x}{6 + 4y}$$

When $\frac{dy}{dx} = \frac{4}{3}$:

$$\frac{4 - 2x}{6 + 4y} = \frac{4}{3}$$

$$12 - 6x = 24 + 16y$$

$$6x + 16y = -12$$

$$3x + 8y = -6$$

$$x = -\frac{6 + 8y}{3}$$

Substituting into the equation of the curve:

$$\left(-\frac{6 + 8y}{3}\right)^2 - 4\left(-\frac{6 + 8y}{3}\right) + 6y + 2y^2 = 12$$

$$\frac{(6 + 8y)^2}{9} + \frac{4(6 + 8y)}{3} + 6y + 2y^2 = 12$$

$$(6 + 8y)^2 + 12(6 + 8y) + 54y + 18y^2 = 108$$

$$64y^2 + 96y + 36 + 72 + 96y + 54y + 18y^2 = 108$$

$$82y^2 + 246y + 108 = 108$$

$$82y^2 + 246y = 0$$

$$y(82y + 246) = 0$$

$y = 0$ or $y = -3$

$x = -\frac{6 + 0}{3} = -2$ $\quad x = -\frac{6 - 24}{3} = 6$

The points are $(-2, 0)$ and $(6, -3)$.

When solving simultaneous equations like in part **b** of worked solution **11**, clear any fractions as soon as possible by multiplying through by the lowest common multiple of all denominators in your working.

12

$$2x + y\ln x = 4y$$

$$2 + y\left(\frac{1}{x}\right) + \frac{dy}{dx}\ln x = 4\frac{dy}{dx}$$

$$\frac{dy}{dx}(4 - \ln x) = 2 + \frac{y}{x} = \frac{2x + y}{x}$$

$$\frac{dy}{dx} = \frac{(2x + y)}{x(4 - \ln x)}$$

At the point $\left(1, \frac{1}{2}\right)$:

$$\frac{dy}{dx} = \frac{2 + \frac{1}{2}}{4 - \ln 1} = \frac{\left(\frac{5}{2}\right)}{4} = \frac{5}{8}$$

Equation of the tangent is:

$$y - \frac{1}{2} = \frac{5}{8}(x - 1)$$

$$y = \frac{5}{8}x - \frac{5}{8} + \frac{1}{2}$$

$$y = \frac{5}{8}x - \frac{1}{8}$$

$$5x - 1 = 8y$$

$$5x - 8y = 1$$

E

13

$$y = x^x$$
$$\ln y = \ln x^x$$
$$\ln y = x \ln x$$
$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \ln x + x\left(\frac{1}{x}\right)$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(\ln x + 1)$$

Stationary points when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$y(\ln x + 1) = 0$

$y = 0$ or $\ln x = -1$

x^x cannot be zero because 0^0 is not defined.

$x = \mathrm{e}^{-1} = \frac{1}{\mathrm{e}}$

14

$$x^2 - xy + y^2 = 48$$
$$2x - x\frac{\mathrm{d}y}{\mathrm{d}x} - y + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$(x - 2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - y$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - y}{x - 2y}$$

Stationary points when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$\frac{2x - y}{x - 2y} = 0$$
$$2x - y = 0$$
$$y = 2x$$

Substituting $y = 2x$ into the equation of the curve:

$$x^2 - x(2x) + (2x)^2 = 48$$
$$3x^2 = 48$$
$$x^2 = 16$$

$x = 4$ or $x = -4$

$y = 2x = 8$ $\quad y = 2x = -8$

The points are $(4, 8)$ and $(-4, -8)$.

Differentiating again:

$$2x - x\frac{\mathrm{d}y}{\mathrm{d}x} - y + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$2 - \frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0$$

At the stationary point $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(2y - x) = -2$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2}{2y - x}$$

At $(4, 8)$:

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2}{16 - 4} = -\frac{1}{6} < 0$, so this is a maximum point.

At $(-4, -8)$:

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2}{-16 + 4} = \frac{1}{6} > 0$, so this is a minimum point.

EXERCISE 4G

1 **c** $x = 2\theta - \sin 2\theta$

$$\frac{dx}{d\theta} = 2 - 2\cos 2\theta$$

$$y = 2 - \cos 2\theta$$

$$\frac{dy}{d\theta} = 2\sin 2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{2\sin 2\theta}{2 - 2\cos 2\theta}$$

$$= \frac{\sin 2\theta}{1 - \cos 2\theta}$$

It is also quite common to see the formula for $\frac{dy}{dx}$, which comes from the chain rule, written as:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}.$$

g $x = 1 + 2\sin^2\theta$

$$\frac{dx}{d\theta} = 4\sin\theta\cos\theta$$

$$y = 4\tan\theta$$

$$\frac{dy}{d\theta} = 4\sec^2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{4\sec^2\theta}{4\sin\theta\cos\theta}$$

$$= \frac{\sec^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\sec^2\theta}{2\sin\theta\cos\theta}$$

$$= \frac{2\sec^2\theta}{\sin 2\theta}$$

2 $x = 3t$

$$\frac{dx}{dt} = 3$$

$$y = t^3 + 4t^2 - 3t$$

$$\frac{dy}{dt} = 3t^2 + 8t - 3$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{3t^2 + 8t - 3}{3}$$

When $\frac{dy}{dx} = 0$:

$$\frac{3t^2 + 8t - 3}{3} = 0$$

$$3t^2 + 8t - 3 = 0$$

$$(3t - 1)(t + 3) = 0$$

$$t = \frac{1}{3} \text{ or } t = -3$$

3 $x = 2\sin\theta$

$$\frac{dx}{d\theta} = 2\cos\theta$$

$$y = 1 - 3\cos 2\theta$$

$$\frac{dy}{d\theta} = 6\sin 2\theta$$

$$\frac{dy}{dx} = \frac{6 \sin 2\theta}{2 \cos \theta} = \frac{12 \sin \theta \cos \theta}{2 \cos \theta} = 6 \sin \theta$$

When $\theta = \frac{\pi}{3}$:

$$\frac{dy}{dx} = 6 \sin \frac{\pi}{3} = 6\frac{\sqrt{3}}{2} = 3\sqrt{3}$$

4 $x = 2 + \ln(t - 1)$

$$\frac{dx}{dt} = \frac{1}{t-1}$$

$$y = t + \frac{4}{t} = t + 4t^{-1}$$

$$\frac{dy}{dt} = 1 - 4t^{-2}$$

$$\frac{dy}{dx} = \frac{1 - 4t^{-2}}{\left(\frac{1}{t-1}\right)} = (t-1)\left(1 - 4t^{-2}\right)$$

$$= t - 4t^{-1} - 1 + 4t^{-2}$$

$$= t - 1 - \frac{4}{t} + \frac{4}{t^2}$$

When $\frac{dy}{dx} = 0$:

$$t - 1 - \frac{4}{t} + \frac{4}{t^2} = 0$$

$$t^3 - t^2 - 4t + 4 = 0$$

$$(1)^3 - (1)^2 - 4(1) + 4 = 0$$

So $(t - 1)$ is a factor of $t^3 - t^2 - 4t + 4$ by the factor theorem.

Dividing by $t - 1$:

$$t^3 - t^2 - 4t + 4 = (t-1)(t^2 + 0t - 4)$$

$$= (t-1)(t-2)(t+2)$$

$$(t-1)(t-2)(t+2) = 0$$

$$t = 1,\ 2 \text{ or } -2$$

but you can only take the logarithm of a positive number, so $t - 1 > 0$.

$t > 1$

i.e. $t = 2$

When $t = 2$:

$$x = 2 + \ln 1 = 2$$

$$y = 2 + \frac{4}{2} = 4$$

The point is $(2,\ 4)$.

> At this stage you will probably have developed a preferred method for dividing polynomials. Use this method to check the division in this worked solution.

5 $x = e^{2t}$

$$\frac{dx}{dt} = 2e^{2t}$$

$$y = 1 + 2te^t$$

$$\frac{dy}{dt} = 2te^t + 2e^t$$

$$\frac{dy}{dx} = \frac{2te^t + 2e^t}{2e^{2t}} = \frac{t+1}{e^t}$$

At $t = 0$:

$$x = e^0 = 1$$

$$y = 1 + 0 = 1$$

$$\frac{dy}{dx} = \frac{0+1}{e^0} = 1$$

Gradient of the normal $= -\frac{1}{1} = -1$

Equation of the normal is:

$$\begin{aligned} y - 1 &= -1(x - 1) \\ y - 1 &= -x + 1 \\ y &= -x + 2 \\ x + y &= 2 \end{aligned}$$

6 a $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{2te^{-t} - t^2e^{-t}}{2e^{2t}}$

$= \frac{2t - t^2}{2e^{3t}}$

$= \frac{t(2 - t)}{2e^{3t}}$

b At the point $(1, -1)$:

$1 = e^{2t}$

$2t = \ln 1 = 0$

$t = 0$

At this point:

$\frac{dy}{dx} = \frac{t(2 - t)}{2e^{3t}} = \frac{0(2 - 0)}{2} = 0$

Gradient is zero, so parallel to the x-axis.

$\frac{t(2 - t)}{2e^{3t}}$ is also zero when $2 - t = 0$.

When $t = 2$:

$x = e^4$

$y = 4e^{-2} - 1$

So the point has coordinates $\left(e^4, \frac{4}{e^2} - 1\right)$.

7 a $\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{4\cos 2\theta}{\sec^2\theta}$

$= 4\cos^2\theta\cos 2\theta$

$= 4\cos^2\theta(2\cos^2\theta - 1)$

b Stationary when $\frac{dy}{dx} = 0$:

$4\cos^2\theta(2\cos^2\theta - 1) = 0$

$\cos\theta = 0$

No solutions in the required range.

Or

$\cos^2\theta = \frac{1}{2}$

$\cos\theta = \pm\frac{1}{\sqrt{2}}$

$\cos\theta = -\frac{1}{\sqrt{2}}$ has no solution in the required range.

$\cos\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$

$x = \tan\frac{\pi}{4} = 1$

$y = 2\sin\left(2 \times \frac{\pi}{4}\right) = 2\sin\left(\frac{\pi}{2}\right) = 2$

The point is $(1, 2)$.

Always check that your solutions are in the range given in the question. Sometimes this is difficult to see, especially if there are lots of square roots or fractions, which can be hard to compare. If that's the case then use your calculator.

8 **a** $x = t + 4\ln t$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + \frac{4}{t} = \frac{t+4}{t}$$

$y = t + 9t^{-1}$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - 9t^{-2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1-9t^{-2}}{\left(\frac{t+4}{t}\right)}$$

$$= \frac{t}{t+4}(1-9t^{-2})$$

$$= \frac{t-9t^{-1}}{t+4}$$

$$= \frac{t^2-9}{t^2+4t}$$

b At the stationary point $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$\frac{t^2-9}{t^2+4t} = 0$$

$t^2 - 9 = 0$

$t^2 = 9$

$t = \pm 3$

$t > 0$

So $t = 3$

$y = 3 + \frac{9}{3} = 6$

$t = 2$ gives $y = 2 + \frac{9}{2} = 6.5 > 6$,

So the stationary point is a minimum.

9 $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{2\sec^2\theta}{4\sin\theta\cos\theta}$$

$$= \frac{1}{2\sin\theta\cos^3\theta}$$

When $\theta = \frac{\pi}{4}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sin\left(\frac{\pi}{4}\right)\cos^3\left(\frac{\pi}{4}\right)}$$

$$= \frac{1}{2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)^3}$$

$$= \frac{8}{4} = 2$$

Gradient of the normal $= -\frac{1}{2}$

When $\theta = \frac{\pi}{4}$:

$$x = 1 + 2\sin^2\frac{\pi}{4} = 1 + 2\left(\frac{\sqrt{2}}{2}\right)^2 = 1 + 1 = 2$$

$$y = 1 + 2\tan\left(\frac{\pi}{4}\right) = 3$$

Equation of the normal is:

$$y - 3 = -\frac{1}{2}(x - 2)$$
$$y - 3 = -\frac{1}{2}x + 1$$
$$y = -\frac{1}{2}x + 4$$

10 **a** $x = 2\sin\theta + \cos 2\theta$

$$\frac{dx}{d\theta} = 2\cos\theta - 2\sin 2\theta$$
$$y = 1 + \cos 2\theta$$
$$\frac{dy}{d\theta} = -2\sin 2\theta$$
$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{-2\sin 2\theta}{2\cos\theta - 2\sin 2\theta}$$
$$= \frac{-4\sin\theta\cos\theta}{2\cos\theta - 4\sin\theta\cos\theta}$$
$$= -\frac{2\sin\theta}{1 - 2\sin\theta}$$
$$= \frac{2\sin\theta}{2\sin\theta - 1}$$

b When $\frac{dy}{dx} = 0$:

$$\frac{2\sin\theta}{2\sin\theta - 1} = 0$$
$$\sin\theta = 0$$

$\theta = 0$ is the only solution in the required range.

$x = 0 + 1 = 1$

$y = 1 + 1 = 2$

The point has coordinates (1, 2).

c At the point $\left(\frac{3}{2}, \frac{3}{2}\right)$:

$$y = \frac{3}{2}$$
$$1 + \cos 2\theta = \frac{3}{2}$$
$$\cos 2\theta = \frac{1}{2}$$

$2\theta = \frac{\pi}{3}$ in the required range.

$\theta = \frac{\pi}{6}$

$$\frac{dy}{dx} = \frac{2\sin\theta}{2\sin\theta - 1}$$

The value of the denominator at $\theta = \frac{\pi}{6}$ is:

$$2\sin\left(\frac{\pi}{6}\right) - 1 = 2\left(\frac{1}{2}\right) - 1 = 0$$

Therefore the tangent is parallel to the y-axis at this point.

11 **a** $x = \ln(\tan t)$

$$\frac{dx}{dt} = \frac{\sec^2 t}{\tan t} = \frac{\cos t}{\sin t\cos^2 t}$$
$$= \frac{1}{\sin t\cos t}$$
$$= \frac{2}{\sin 2t}$$

$$y = 2\sin 2t$$
$$\frac{dy}{dt} = 4\cos 2t$$
$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{4\cos 2t}{\left(\frac{2}{\sin 2t}\right)}$$
$$= 2\cos 2t \sin 2t$$
$$= \sin 4t$$

b When $x = 0$:

$$\ln(\tan t) = 0$$
$$\tan t = e^0 = 1$$
$$t = \frac{\pi}{4}$$
$$\frac{dy}{dx} = \sin\left(4 \times \frac{\pi}{4}\right) = \sin \pi = 0$$

Gradient is zero, so the tangent is parallel to the x-axis.

END-OF-CHAPTER REVIEW EXERCISE 4

1 **i** $x = 1 + \ln(t - 2)$

$$\frac{dx}{dt} = \frac{1}{t-2}$$

$$y = t + 9t^{-1}$$

$$\frac{dy}{dt} = 1 - 9t^{-2} = 1 - \frac{9}{t^2}$$

$$\frac{dy}{dx} = \frac{1 - \frac{9}{t^2}}{\left(\frac{1}{t-2}\right)} = (t-2)\left(1 - \frac{9}{t^2}\right)$$

$$= (t-2)\left(\frac{t^2 - 9}{t^2}\right)$$

$$= \frac{(t^2 - 9)(t - 2)}{t^2}$$

ii When $\frac{dy}{dx} = 0$:

$$\frac{(t^2 - 9)(t - 2)}{t^2} = 0$$

$t^2 = 9 \quad \Rightarrow \quad t = \pm 3$

or $\quad t = 2$

$t > 2$ (given in the question)

$t = 3$ is the only solution.

$x = 1 + \ln(3 - 2) = 1$

$y = 3 + \frac{9}{3} = 6$

The point is $(1, 6)$.

When it appears that there are multiple solutions, check that they are all valid. Conditions may have been written into the question, as in Question **1**, or you may not be able to substitute a solution into the original equation.

2 **i** $y = x\ln(x - 3)$

$$\frac{dy}{dx} = x\left(\frac{1}{x-3}\right) + \ln(x - 3)$$

$$= \frac{x}{x-3} + \ln(x - 3)$$

When $x = 4$:

$$\frac{dy}{dx} = \frac{4}{1} + \ln 1 = 4$$

ii $y = \frac{x-1}{x+1}$

$$\frac{dy}{dx} = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2}$$

$$= \frac{x + 1 - x + 1}{(x+1)^2}$$

$$= \frac{2}{(x+1)^2}$$

When $x = 4$:

$$\frac{dy}{dx} = \frac{2}{5^2} = \frac{2}{25}$$

3 **i** $x = e^{3t}$

$\frac{dx}{dt} = 3e^{3t}$

$y = t^2 e^t + 3$

$\frac{dy}{dt} = t^2 e^t + 2te^t$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{t^2 e^t + 2te^t}{3e^{3t}}$

$= \frac{t^2 + 2t}{3e^{2t}}$

$= \frac{t(t+2)}{3e^{2t}}$

ii When $x = 1$:

$e^{3t} = 1$

$t = 0$

$\frac{dy}{dx} = \frac{0(0+2)}{3e^0} = 0$

The tangent is parallel to the x-axis.

iii When $\frac{dy}{dx} = 0$:

$\frac{t(t+2)}{3e^{2t}} = 0$

Other solution is $t = -2$.

When $t = -2$:

$x = e^{3(-2)} = e^{-6}$

$y = (-2)^2 e^{-2} + 3 = 4e^{-2} + 3$

The point has coordinates $(e^{-6}, 4e^{-2} + 3)$.

4 **i** $y = 3\sin x + \tan 2x$

$\frac{dy}{dx} = 3\cos x + 2\sec^2 2x$

When $x = 0$:

$\frac{dy}{dx} = 3\cos 0 + \frac{2}{\cos^2 0}$

$= 3 + 2 = 5$

ii $y = \frac{6}{1 + e^{2x}} = 6\left(1 + e^{2x}\right)^{-1}$

$\frac{dy}{dx} = -6\left(1 + e^{2x}\right)^{-2}(2e^{2x})$

$= -\frac{12e^{2x}}{(1 + e^{2x})^2}$

When $x = 0$:

$\frac{dy}{dx} = -\frac{12e^0}{(1 + e^0)^2}$

$= -\frac{12}{2^2} = -3$

5 **i** $2x^2 + 3xy + y^2 = 3$

$4x + 3x\frac{dy}{dx} + 3y + 2y\frac{dy}{dx} = 0$

$(3x + 2y)\frac{dy}{dx} = -3y - 4x$

$\frac{dy}{dx} = -\frac{3y + 4x}{3x + 2y}$

When $x = 2$ and $y = -1$:

$\frac{dy}{dx} = -\frac{-3 + 8}{6 - 2} = -\frac{5}{4}$

Equation of the tangent is:

$$y-(-1)=-\frac{5}{4}(x-2)$$
$$y=-\frac{5}{4}x+\frac{5}{2}-1$$
$$4y=-5x+10-4$$
$$5x+4y-6=0$$

ii If the curve has stationary points then $\frac{dy}{dx}=0$.

$$-\frac{3y+4x}{3x+2y}=0$$
$$4x+3y=0$$
$$x=-\frac{3y}{4}$$

Substituting $x=-\frac{3y}{4}$ into the equation of the curve:

$$2x^2+3xy+y^2=3$$
$$2\left(-\frac{3y}{4}\right)^2+3y\left(-\frac{3y}{4}\right)+y^2=3$$
$$\frac{9y^2}{8}-\frac{9y^2}{4}+y^2=3$$
$$18y^2-36y^2+16y^2=48$$
$$-2y^2=48$$
$$y^2=-\frac{48}{2}$$

But $y^2 \geqslant 0$

so there are no solutions.

The curve has no stationary points.

6 i

$$y^3+4xy=16$$
$$3y^2\frac{dy}{dx}+4x\frac{dy}{dx}+4y=0$$
$$(3y^2+4x)\frac{dy}{dx}=-4y$$
$$\frac{dy}{dx}=-\frac{4y}{3y^2+4x}$$

ii If there are stationary points then $\frac{dy}{dx}=0$:

$$-\frac{4y}{3y^2+4x}=0$$
$$y=0$$

Substituting $y=0$ into the equation of the curve: $0+0=16$, which is not possible.

There are no stationary points on this curve.

iii The tangent is parallel to the y-axis when the denominator of $-\frac{4y}{3y^2+4x}$ is zero.

$$3y^2+4x=0$$
$$4x=-3y^2$$
$$x=-\frac{3y^2}{4}$$

Substituting $x=-\frac{3y^2}{4}$ into the equation of the curve:

$$y^3 + 4\left(-\frac{3y^2}{4}\right)y = 16$$
$$y^3 - 3y^3 = 16$$
$$-2y^3 = 16$$
$$y^3 = -8$$
$$y = -2$$
$$x = -\frac{3(-2)^2}{4} = -3$$

The point has coordinates $(-3, -2)$.

7 $y = 6\sin x - 2\cos 2x$

$$\frac{dy}{dx} = 6\cos x + 4\sin 2x$$

When $x = \frac{\pi}{6}$:

$$\frac{dy}{dx} = 6\cos\frac{\pi}{6} + 4\sin\frac{\pi}{3} = \frac{6\sqrt{3}}{2} + \frac{4\sqrt{3}}{2} = \frac{10\sqrt{3}}{2}$$

Equation of the tangent is:

$$y - 2 = 5\sqrt{3}(x - \frac{\pi}{6})$$
$$y = 5\sqrt{3}x - 5\sqrt{3}\left(\frac{\pi}{6}\right) + 2$$

$y = 8.66x - 2.53$ (to 3 significant figures)

Only use decimals if the question says that you should, as was the case in Question **7**.

8 **i** $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{4\cos 2t - 6\sin 2t}{12\sin t\cos t}$

$$= \frac{4\cos 2t - 6\sin 2t}{6(2\sin t\cos t)}$$
$$= \frac{4\cos 2t - 6\sin 2t}{6\sin 2t}$$
$$= \frac{2\cos 2t}{3\sin 2t} - 1$$
$$= \frac{2}{3}\cot 2t - 1$$

ii When $\frac{dy}{dx} = 0$:

$$\frac{2}{3}\cot 2t - 1 = 0$$
$$\cot 2t = \frac{3}{2}$$
$$\tan 2t = \frac{2}{3}$$

$2t = 0.58800...$ or $2t = 0.58800... + \pi = 3.72959...$

$t = 0.294$ $\quad$ $t = 1.865$ (to 3 decimal places)

iii At B, $y = 0$:

$$2\sin 2t + 3\cos 2t = 0$$
$$2\sin 2t = -3\cos 2t$$
$$\frac{\sin 2t}{\cos 2t} = -\frac{3}{2}$$
$$\tan 2t = -\frac{3}{2}$$

When $\tan 2t = -\frac{3}{2}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{3}\cot 2t - 1$$
$$= \frac{2}{3\tan 2t} - 1$$
$$= \frac{2}{3\left(-\frac{3}{2}\right)} - 1$$
$$= -\frac{4}{9} - 1$$
$$= -\frac{13}{9}$$

9
$$3x^2 + 4xy + y^2 = 24$$
$$6x + 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$(4x + 2y)\frac{\mathrm{d}y}{\mathrm{d}x} = -4y - 6x$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4y + 6x}{4x + 2y} = -\frac{2y + 3x}{2x + y}$$

At the point $(1,\ 3)$ when $x = 1$ and $y = 3$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{6 + 3}{2 + 3} = -\frac{9}{5}$$

Gradient of the normal $= -\frac{1}{\left(-\frac{9}{5}\right)} = \frac{5}{9}$

Equation of the normal is:

$$y - 3 = \frac{5}{9}(x - 1)$$
$$9y - 27 = 5x - 5$$
$$5x - 9y + 22 = 0$$

10 **i**
$$y = \frac{1 - x}{1 + x}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1 + x)(-1) - (1 - x)(1)}{(1 + x)^2}$$
$$= \frac{-1 - x - 1 + x}{(1 + x)^2}$$
$$= -\frac{2}{(1 + x)^2}$$

$$y = \sqrt{\frac{1 - x}{1 + x}} = \left(\frac{1 - x}{1 + x}\right)^{\frac{1}{2}}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(\frac{1 - x}{1 + x}\right)^{-\frac{1}{2}}\left(-\frac{2}{(1 + x)^2}\right)$$
$$= -\left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}}\left(\frac{1}{(1 + x)^2}\right)$$
$$= -\frac{(1 + x)^{\frac{1}{2}}}{(1 - x)^{\frac{1}{2}}(1 + x)^2}$$

Gradient of the normal at the point (x, y)

$$= -\frac{1}{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}$$
$$= \frac{(1-x)^{\frac{1}{2}}(1+x)^2}{(1+x)^{\frac{1}{2}}}$$
$$= (1-x)^{\frac{1}{2}}(1+x)^{2-\frac{1}{2}}$$
$$= (1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}$$
$$= (1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}(1+x)^1$$
$$= [(1-x)\,(1+x)]^{\frac{1}{2}}(1+x)$$
$$= (1+x)\sqrt{(1-x^2)}$$

> Notice that when you take the reciprocal of a fraction (as you do when you raise it to the power of -1), then it simply turns the fraction 'upside down'.

ii $y = (1+x)(1-x^2)^{\frac{1}{2}}$

This has a maximum point when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)(1+x) + \left(1-x^2\right)^{\frac{1}{2}}$$
$$= \frac{-2x(1+x)}{2\sqrt{1-x^2}} + \sqrt{1-x^2}$$
$$= \frac{-2x(1+x) + 2(1-x^2)}{2\sqrt{1-x^2}}$$
$$= \frac{-2x - 2x^2 + 2 - 2x^2}{2\sqrt{1-x^2}}$$
$$= \frac{-4x^2 - 2x + 2}{2\sqrt{1-x^2}}$$

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$-4x^2 - 2x + 2 = 0$$
$$2x^2 + x - 1 = 0$$
$$(2x-1)(x+1) = 0$$
$$x = \frac{1}{2} \quad \text{or} \quad x = -1$$

Given that the x-coordinate of P is positive, as seen in the diagram, $x = \frac{1}{2}$

11 $x = 3(1+\sin^2 t) = 3 + 3\sin^2 t$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 6\sin t\cos t$$
$$y = 2\cos^3 t$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 6\cos^2 t(-\sin t) = -6\sin t\cos^2 t$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{-6\sin t\cos^2 t}{6\sin t\cos t} = -\cos t$$

12 **i**

$$\ln(xy) - y^3 = 1$$
$$\frac{1}{xy}\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) - 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x} - 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{1}{y} - 3y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x}$$
$$\left(\frac{1-3y^3}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x}\left(\frac{y}{1-3y^3}\right)$$
$$= -\frac{y}{x(1-3y^3)} = \frac{y}{x\,(3y^3-1)}$$

ii Tangent is parallel to the y-axis when the denominator of $\dfrac{y}{x(3y^3-1)}$ is zero.

$x(3y^3 - 1) = 0$

$x = 0$ or $3y^3 - 1 = 0$

The original equation is not defined for $x = 0$, so:

$$3y^3 - 1 = 0$$
$$3y^3 = 1$$
$$y^3 = \frac{1}{3}$$

$y = \sqrt[3]{\left(\frac{1}{3}\right)} = 0.693$ (to 3 significant figures)

Returning to the equation of the curve for x:

$$\ln(xy) - y^3 = 1$$
$$\ln(xy) = 1 + y^3$$
$$xy = \mathrm{e}^{1+y^3}$$

$x = \dfrac{\mathrm{e}^{1+y^3}}{y} = \dfrac{\mathrm{e}^{1+\frac{1}{3}}}{\sqrt[3]{\left(\frac{1}{3}\right)}} = 5.47$ (to 3 significant figures)

The point is $(5.47,\ 0.693)$ (to 3 significant figures)

13 **i** $6\mathrm{e}^{2x} + k\mathrm{e}^{y} + \mathrm{e}^{2y} = c$

$y = \ln 2$ when $x = \ln 3$:

$$6\mathrm{e}^{2\ln 3} + k\mathrm{e}^{\ln 2} + \mathrm{e}^{2\ln 2} = c$$
$$6\mathrm{e}^{\ln 9} + 2k + \mathrm{e}^{\ln 4} = c$$
$$6 \times 9 + 2k + 4 = c$$
$$58 + 2k = c$$

ii Differentiating:

$$12\mathrm{e}^{2x} + k\mathrm{e}^{y}\frac{\mathrm{d}y}{\mathrm{d}x} + 2\mathrm{e}^{2y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$(k\mathrm{e}^{y} + 2\mathrm{e}^{2y})\frac{\mathrm{d}y}{\mathrm{d}x} = -12\mathrm{e}^{2x}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{12\mathrm{e}^{2x}}{(k\mathrm{e}^{y} + 2\mathrm{e}^{2y})}$$

Using the fact that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -6$ at P:

$$
\begin{aligned}
-\frac{12e^{2\ln 3}}{(ke^{\ln 2} + 2e^{2\ln 2})} &= -6 \\
\frac{12e^{\ln 9}}{2k + 2e^{\ln 4}} &= 6 \\
\frac{12 \times 9}{2k + 2 \times 4} &= 6 \\
108 &= 12k + 48 \\
12k &= 60 \\
k &= 5
\end{aligned}
$$

But $58 + 2k = c$ from part **i**, so:

$$
\begin{aligned}
58 + 10 &= c \\
c &= 68
\end{aligned}
$$

Chapter 5
Integration

EXERCISE 5A

1 Remember that you can take a factor out of any integral. These worked solutions show how this can help when finding integrals.

c $$\int 6e^{3x}\,dx = 2\int 3e^{3x}\,dx$$
$$= 2e^{3x} + c$$

h $$\int 6e^{2-3x}\,dx = -2\int -3e^{2-3x}\,dx$$
$$= -2e^{2-3x} + c$$

2 b $$\int 5e^{x}(2+e^{3x})\,dx = \int 10e^{x} + 5e^{4x}\,dx$$
$$= 10e^{x} + \frac{5}{4}e^{4x} + c$$
$$= \frac{5}{4}e^{x}(e^{3x}+8) + c$$

d $$\int \frac{4+e^{2x}}{e^{2x}}\,dx = \int \left(\frac{4}{e^{2x}}+1\right)dx$$
$$\int (4e^{-2x}+1)\,dx = -2e^{-2x} + x + c$$

3 c $$\int_0^{\ln 2} 5e^{-2x}\,dx = \left[-\frac{5}{2}e^{-2x}\right]_0^{\ln 2}$$
$$= -\frac{5}{2}e^{-2\ln 2} - \left(-\frac{5}{2}e^{-2(0)}\right)$$
$$= -\frac{5}{2}e^{\ln(2^{-2})} + \frac{5}{2}e^{0}$$
$$= -\frac{5}{2}\left(\frac{1}{4}\right) + \frac{5}{2}$$
$$= -\frac{5}{8} + \frac{20}{8} = \frac{15}{8}$$

g $$\int_0^1 (e^{x}+e^{2x})^2\,dx$$
$$= \int_0^1 \left[(e^{x})^2 + 2e^{x}e^{2x} + (e^{2x})^2\right]dx$$
$$= \int_0^1 (e^{2x} + 2e^{3x} + e^{4x})\,dx$$
$$= \left[\frac{1}{2}e^{2x} + \frac{2}{3}e^{3x} + \frac{1}{4}e^{4x}\right]_0^1$$
$$= \frac{1}{2}e^{2} + \frac{2}{3}e^{3} + \frac{1}{4}e^{4} - \frac{1}{2}e^{0} - \frac{2}{3}e^{0} - \frac{1}{4}e^{0}$$
$$= \frac{1}{2}e^{2} + \frac{2}{3}e^{3} + \frac{1}{4}e^{4} - \frac{1}{2} - \frac{2}{3} - \frac{1}{4}$$
$$= \frac{1}{12}(6e^{2} + 8e^{3} + 3e^{4} - 6 - 8 - 3)$$
$$= \frac{1}{12}(6e^{2} + 8e^{3} + 3e^{4} - 17)$$

4 $\frac{dy}{dx} = 6e^{2x} + 2e^{-x}$

Integrating:

$$y = \int \left(6e^{2x} + 2e^{-x}\right) dx$$
$$= 3e^{2x} - 2e^{-x} + c$$

Using the fact that $y = 2$ when $x = 0$:

$$2 = 3 - 2 + c$$
$$c = 1$$
$$y = 3e^{2x} - 2e^{-x} + 1$$

> These are examples of differential equations, where you are given the outcome after differentiation and asked to work backwards to the original function. There are many types of differential equation; this is the simplest. The solution to a differential equation is always a function.

5 $\frac{d^2y}{dx^2} = 20e^{-2x}$

Integrating:

$$\frac{dy}{dx} = -10e^{-2x} + c$$

Using the fact that $\frac{dy}{dx} = -8$ when $x = 0$:

$$-8 = -10e^{0} + c$$
$$c = -8 + 10 = 2$$
$$\frac{dy}{dx} = -10e^{-2x} + 2$$

Integrating:

$$y = 5e^{-2x} + 2x + d$$

Using the fact that $y = \frac{5}{e^2}$ when $x = 1$:

$$\frac{5}{e^2} = \frac{5}{e^2} + 2 + d$$
$$d = -2$$
$$y = 5e^{-x} + 2x - 2$$

6
$$\int_1^3 \left(1 + e^{2x-5}\right) dx = \left[x + \frac{1}{2}e^{2x-5}\right]_1^3$$
$$= 3 + \frac{1}{2}e^{6-5} - \left(1 + \frac{1}{2}e^{2-5}\right)$$
$$= 3 + \frac{1}{2}e - 1 - \frac{1}{2}e^{-3}$$
$$= 2 + \frac{1}{2}e - \frac{1}{2e^3}$$
$$= \frac{1}{2}\left(4 + e - \frac{1}{e^3}\right)$$

7
$$\int_0^3 \left(2e^{\frac{1}{2}x} - 2x + 3\right) dx = \left[4e^{\frac{1}{2}x} - x^2 + 3x\right]_0^3$$
$$= 4e^{\frac{3}{2}} - 9 + 9 - \left(4e^0 - 0 + 0\right)$$
$$= 4e^{\frac{3}{2}} - 4$$
$$= 4(e^{\frac{3}{2}} - 1)$$

8 **a**
$$\frac{d}{dx}(xe^x - e^x) = xe^x + e^x - e^x$$
$$= xe^x$$

b $\int_0^3 xe^x \, dx = [xe^x - e^x]_0^3$

$= 3e^3 - e^3 - (0 - e^0)$

$= 2e^3 + 1$

9 **a** $\int_0^a \left(4e^{-2x} + 5e^{-x}\right) dx = \left[-2e^{-2x} - 5e^{-x}\right]_0^a$

$= -2e^{-2a} - 5e^{-a} - (-2e^0 - 5e^0)$

$= 7 - \frac{2}{e^{2a}} - \frac{5}{e^a}$

b $\int_0^\infty \left(4e^{-2x} + 5e^{-x}\right) dx = \lim_{a\to\infty} \int_0^a \left(4e^{-2x} + 5e^{-x}\right) dx$

$= \lim_{a\to\infty} \left(7 - \frac{2}{e^{2a}} - \frac{5}{e^a}\right)$

$= 7 - 0 - 0$

$= 7$

Limit notation, which you use when you let a variable increase without limit, is used in Worked solution **9b**. As a gets bigger and bigger, both $\frac{2}{e^{2a}}$ and $\frac{5}{e^a}$ get smaller and smaller. We say that they tend to zero.

10 Setting the derivative equal to zero to find the x-coordinate of the minimum point M:

$y = 2e^x + 8e^{-x} - 7$

$\frac{dy}{dx} = 2e^x - 8e^{-x}$

Using the fact that at M, $\frac{dy}{dx} = 0$:

$2e^x - 8e^{-x} = 0$

$2e^x - \frac{8}{e^x} = 0$

$2(e^x)^2 - 8 = 0$

$(e^x)^2 = 4$

$e^x = \pm\sqrt{4} = \pm 2$

$e^x > 0$

So $e^x = 2$

$e^x = 2$

Now using the x-coordinate of point M as the upper limit of an integral:

$\int_0^{\ln 2} (2e^x + 8e^{-x} - 7) \, dx$

$= [2e^x - 8e^{-x} - 7x]_0^{\ln 2}$

$= 2e^{\ln 2} - 8e^{-\ln 2} - 7\ln 2 - (2 - 8 - 0)$

$= 2(2) - 8e^{\ln 2^{-1}} - 7\ln 2 + 6$

$= 10 - 8\left(\frac{1}{2}\right) - 7\ln 2$

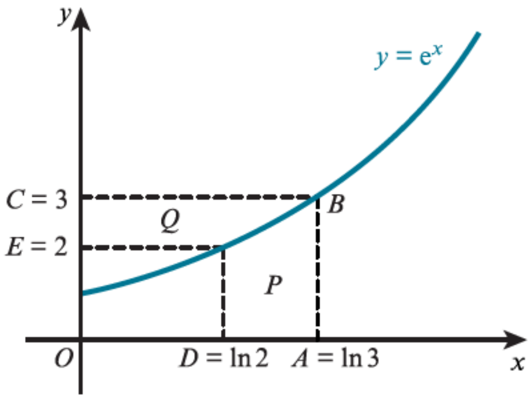
$= 6 - 7\ln 2$

11 **a** $\int_{\ln 2}^{\ln 3} e^x \, dx = [e^x]_{\ln 2}^{\ln 3}$

$= e^{\ln 3} - e^{\ln 2}$

$= 3 - 2 = 1$

b



The integral, from part **a**, represents the area of P in the diagram.

If $y = e^x$ then $x = \ln\ y$, so:

$\int_2^3 \ln y\, dy$ represents the area Q in the diagram.

$Q = \int_2^3 \ln y\, dy = \text{Area of rectangle}$

$OABC - P - \text{Area of rectangle } DFEO$

$= 3\ln\ 3 - 1 - 2\ln\ 2$

$= \ln\ 27 - \ln\ e - \ln\ 4$

$= \ln\left(\frac{27}{4e}\right)$

EXERCISE 5B

1

When the numerator is the derivative of the denominator, you can use logarithms, as described in the coursebook. These worked solutions show how you can take out factors to adjust the numerator if it is almost the derivative you need.

e $$\int \frac{5}{2-3x}\,dx = -\frac{5}{3}\int \frac{-3}{2-3x}\,dx$$
$$= -\frac{5}{3}\ln|2-3x| + c$$

f $$\int \frac{3}{2(5x-1)}\,dx = \frac{3}{2}\int \frac{1}{5x-1}\,dx$$
$$= \frac{3}{10}\int \frac{5}{5x-1}\,dx$$
$$= \frac{3}{10}\ln|5x-1| + c$$

2 **b** $$\int_1^4 \frac{1}{2x+1}\,dx = \frac{1}{2}\int_1^4 \frac{2}{2x+1}\,dx$$
$$= \frac{1}{2}[\ln|2x+1|]_1^4$$
$$= \frac{1}{2}(\ln 9 - \ln 3)$$
$$= \frac{1}{2}\ln\left(\frac{9}{3}\right)$$
$$= \frac{1}{2}\ln 3$$

f $$\int_2^3 \frac{4}{3-2x}\,dx = -2\int_2^3 \frac{-2}{3-2x}\,dx$$
$$= -2[\ln|3-2x|]_2^3$$
$$= -2\left(\ln|-3| - \ln|-1|\right)$$
$$= -2(\ln 3 - \ln 1)$$
$$= -2\ln 3$$
$$= -\ln 3^2$$
$$= -\ln 9$$

Don't forget that you need to use a modulus function here. In worked solution **2f** it made a difference.

3 **a** $$\int_4^{10}\left(2 + \frac{5}{3x-2}\right)dx$$
$$= \left[2x + \frac{5}{3}\ln|3x-2|\right]_4^{10}$$
$$= 20 + \frac{5}{3}\ln 28 - \left(8 + \frac{5}{3}\ln 10\right)$$
$$= 12 + \frac{5}{3}\ln\left(\frac{28}{10}\right)$$
$$= 12 + \frac{5}{3}\ln\left(\frac{14}{5}\right)$$

b $\int_1^3 \left(\frac{3}{x} - \frac{4}{2x-1}\right) dx$

$= [3\ln|x| - 2\ln|2x-1|]_1^3$

$= 3\ln 3 - 2\ln 5 - (3\ln 1 - 2\ln 1)$

$= \ln 3^3 - \ln 5^2 - 0 + 0$

$= \ln 27 - \ln 25$

$= \ln\left(\frac{27}{25}\right)$

c $\int_{-1}^3 \left(2x - 1 + \frac{4}{2x+3}\right) dx$

$= \left[x^2 - x + 2\ln|2x+3|\right]_{-1}^3$

$= 9 - 3 + 2\ln 9 - (1 + 1 + 2\ln 1)$

$= 6 + \ln 9^2 - 2 - 0$

$= 4 + \ln 81$

4 **a** $\frac{4x}{2x-1} = \frac{2(2x-1)+2}{2x-1}$

$= 2 + \frac{2}{2x-1}$

$A = 2$

b $\int_1^5 \frac{4x}{2x-1} dx = \int_1^5 \left(2 + \frac{2}{2x-1}\right) dx$

$= [2x + \ln|2x-1|]_1^5$

$= 10 + \ln 9 - (2 + \ln 1)$

$= 8 + \ln 9$

5 **a** Using long division:

$$\begin{array}{r} 3x + 10 \\ 2x - 5\overline{)6x^2 + 5x} \\ \underline{6x^2 - 15x} \\ 20x \\ \underline{20x - 50} \\ 50 \end{array}$$

Quotient $= 3x + 10$

Remainder $= 50$

b $\frac{6x^2 + 5x}{2x-5} = 3x + 10 + \frac{50}{2x-5}$

$\int_0^1 \left(3x + 10 + \frac{50}{2x-5}\right) dx$

$= \left[\frac{3x^2}{2} + 10x + 25\ln|2x-5|\right]_0^1$

$= \frac{3}{2} + 10 + 25\ln|-3| - (0 + 0 + 25\ln|-5|)$

$= \frac{23}{2} + 25\ln 3 - 25\ln 5$

$= \frac{23}{2} + 25\ln\left(\frac{5}{3}\right)^{-1}$

$= \frac{23}{2} - 25\ln\left(\frac{5}{3}\right)$

6 $\frac{dy}{dx} = 2x + \frac{3}{x+e}$

$y = \int \left(2x + \frac{3}{x+e}\right) dx$

$= x^2 + 3\ln|x+e| + c$

Using the fact that $y = e^2$ when $x = e$:

$$e^2 = e^2 + 3\ln 2e + c$$
$$c = -3\ln 2e$$
$$y = x^2 + 3\ln|x + e| - 3\ln 2e$$

Remember that 'e' is just a constant; it doesn't complicate the algebra in any way.

7

$$\int_1^k \frac{2}{x+3}\,dx = 4$$
$$[2\ln|x+3|]_1^k = 4$$
$$2\ln(k+3) - 2\ln 4 = 4$$
$$2\ln(k+3) = 4 + 2\ln 4$$
$$\ln(k+3) = 2 + \ln 4$$
$$k + 3 = e^{2+\ln 4}$$
$$k = -3 + e^2 e^{\ln 4}$$
$$k = 4e^2 - 3$$

8

$$\frac{dy}{dx} = 3 - \frac{2}{x}$$
$$y = \int\left(3 - \frac{2}{x}\right)dx$$
$$y = 3x - 2\ln|x| + c$$

Using the fact that $y = -2$ when $x = 1$:

$$-2 = 3 - 2\ln 1 + c$$
$$c = -5$$
$$y = 2x - 2\ln|x| - 5$$

Using the fact that $y = k$ when $x = 2$:

$k = 4 - 2\ln 2 - 5 = -1 - 2\ln 2$

Gradient of tangent at P is:

$$\frac{dy}{dx} = 3 - \frac{2}{1} = 1$$

Equation of the tangent at P is:

$y - (-2) = 1(x - 1)$

$y = x - 3$ .... [1]

Gradient of the tangent at Q is:

$$\frac{dy}{dx} = 3 - \frac{2}{2} = 2$$

Equation of the tangent at Q is:

$y - (-1 - 2\ln 2) = 2(x - 2)$

$y = 2x - 2\ln 2 - 3$ .... [2]

Substituting [1] into [2]:

$$x - 3 = 2x - 2\ln 2 - 3$$
$$x = 2\ln 2$$
$$y = 2\ln 2 - 3$$

So the point is $(2\ln 2,\ 2\ln 2 - 3)$.

EXERCISE 5C

1

Remember to ask yourself the question: "What kind of function differentiates to give the function I am trying to integrate?". Then take this function and differentiate it. How close does that get you?

d $$\int 3\sin 2x\,\mathrm{d}x = -\frac{3}{2}\int -2\sin 2x\,\mathrm{d}x$$
$$= -\frac{3}{2}\cos 2x + c$$

i $$\int 2\sec^2(5x-2)\,\mathrm{d}x = \frac{2}{5}\int 5\sec^2(5x-2)\,\mathrm{d}x$$
$$= \frac{2}{5}\tan(5x-2) + c$$

2 **a** $$\int_0^{\frac{1}{6}\pi} \cos 4x\,\mathrm{d}x = \frac{1}{4}\int_0^{\frac{1}{6}\pi} 4\cos 4x\,\mathrm{d}x$$
$$= \frac{1}{4}[\sin 4x]_0^{\frac{1}{6}\pi}$$
$$= \frac{1}{4}\sin\left(\frac{2\pi}{3}\right) - \frac{1}{4}\sin 0$$
$$= \frac{1}{4}\frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{3}}{8}$$

c $$\int_0^{\frac{\pi}{6}} \sec^2 2x\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{6}} 2\sec^2 2x\,\mathrm{d}x$$
$$= \frac{1}{2}[\tan 2x]_0^{\frac{\pi}{6}}$$
$$= \frac{1}{2}\left(\tan\frac{\pi}{3} - \tan 0\right)$$
$$= \frac{\sqrt{3}}{2}$$

3 **a** $$\frac{\mathrm{d}}{\mathrm{d}x}(x\sin x + \cos x) = x\cos x + \sin x + (-\sin x)$$
$$= x\cos x$$

b Using the result from part **a**:

$$\int_0^{\frac{\pi}{3}} x\cos x\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} \frac{\mathrm{d}}{\mathrm{d}x}(x\sin x + \cos x)\,\mathrm{d}x$$
$$= [x\sin x + \cos x]_0^{\frac{\pi}{3}}$$
$$= \left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right) - 0 - \cos 0$$
$$= \frac{\pi}{3}\frac{\sqrt{3}}{2} + \frac{1}{2} - 1$$
$$= \frac{\pi\sqrt{3}}{6} - \frac{1}{2}$$
$$= \frac{1}{6}(\pi\sqrt{3} - 3)$$

Your work on questions like these will be faster and more accurate if you memorise standard trigonometric values early in your course of study.

4 $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - 3\sin 2x$$

Integrating:

$y = \int (1 - 3\sin 2x)\,dx$

$= x + \frac{3}{2}\cos 2x + c$

Using the fact that $y = 0$ when $x = \frac{\pi}{4}$:

$0 = \frac{\pi}{4} + \frac{3}{2}\cos\left(\frac{2\pi}{4}\right) + c$

$c = -\frac{\pi}{4} - 0$

$y = x + \frac{3}{2}\cos 2x - \frac{\pi}{4}$

5 $\frac{d^2y}{dx^2} = -12\sin 2x - 2\cos x$

Integrating:

$\frac{dy}{dx} = 6\cos 2x - 2\sin x + c$

Using the fact that $\frac{dy}{dx} = 4$ when $x = 0$:

$4 = 6\cos 0 - 2\sin 0 + c$

$c = 4 - 6 = -2$

$\frac{dy}{dx} = 6\cos 2x - 2\sin x - 2$

Integrating:

$y = 3\sin 2x + 2\cos x - 2x + d$

Using the fact that $y = -3$ when $x = \frac{\pi}{2}$:

$-3 = 3\sin\pi + 2\cos\frac{\pi}{2} - \pi + d$

$d = -3 - 0 - 0 + \pi$

$y = 3\sin 2x + 2\cos x - 2x - 3 + \pi$

> Two different constants of integration has been used in worked solution **5**, although there is not really anything wrong with using '*c*' twice. You just have to remember that *c* will have different values in the different parts of the question.

6 **a** $\frac{dy}{dx} = 4\sin\left(2x - \frac{\pi}{2}\right)$

Integrating:

$y = -2\cos\left(2x - \frac{\pi}{2}\right) + c$

Using the fact that $y = 5$ when $x = \frac{\pi}{2}$:

$5 = -2\cos\left(\pi - \frac{\pi}{2}\right) + c$

$5 = -2\cos\frac{\pi}{2} + c = 0 + c$

$c = 5$

$y = -2\cos\left(2x - \frac{\pi}{2}\right) + 5$

b $y = -2\cos\left(2x - \frac{\pi}{2}\right) + 5$

When $x = \frac{\pi}{3}$:

$y = -2\cos\left(2\left(\frac{\pi}{3}\right) - \frac{\pi}{2}\right) + 5$

$y = -2\cos\left(\frac{\pi}{6}\right) + 5 = -2\frac{\sqrt{3}}{2} + 5 = -\sqrt{3} + 5$

$\frac{dy}{dx} = 4\sin\left(\frac{\pi}{6}\right) = 4\left(\frac{1}{2}\right) = 2$

Gradient of normal $= -\frac{1}{2}$

Equation of the normal is:

$$y - (-\sqrt{3} + 5) = -\frac{1}{2}\left(x - \frac{\pi}{3}\right)$$
$$y = -\frac{1}{2}x + \frac{\pi}{6} - \sqrt{3} + 5$$
$$x + 2y = 10 + \frac{\pi}{3} - 2\sqrt{3}$$

7 First, establishing the limits of the integral by finding the points at which the curve crosses the x-axis:

$$1 + \sqrt{3}\sin 2x + \cos 2x = 0$$
$$\sqrt{3}\sin 2x + \cos 2x = R\cos(2x - \alpha)$$
$$\sqrt{3}\sin 2x + \cos 2x = R\cos 2x \cos\alpha + R\sin 2x \sin\alpha$$

Equating coefficients of $\cos 2x$: $R\cos\alpha = 1$ .... [1]

Equating coefficients of $\sin 2x$: $R\sin\alpha = \sqrt{3}$ .... [2]

[2] ÷ [1]:

$$\tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$$
$$R^2 = 1^2 + \left(\sqrt{3}\right)^2 = 1 + 3 = 4$$
$$R = 2$$
$$\sqrt{3}\sin 2x + \cos 2x = 2\cos\left(2x - \frac{\pi}{3}\right)$$
$$1 + \sqrt{3}\sin 2x + \cos 2x = 0$$
$$1 + 2\cos\left(2x - \frac{\pi}{3}\right) = 0$$
$$\cos\left(2x - \frac{\pi}{3}\right) = -\frac{1}{2}$$
$$2x - \frac{\pi}{3} = \frac{2\pi}{3}$$
$$2x = \pi$$
$$x = \frac{\pi}{2}$$

Now, integrating the function between $x = 0$ and $x = \frac{\pi}{2}$:

$$\int_0^{\frac{\pi}{2}} (1 + \sqrt{3}\sin 2x + \cos 2x)\,\mathrm{d}x$$
$$= \left[x - \frac{\sqrt{3}}{2}\cos 2x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{2} - \frac{\sqrt{3}}{2}\cos\pi + \frac{1}{2}\sin\pi - \left(0 - \frac{\sqrt{3}}{2}\cos 0 + \frac{1}{2}\sin 0\right)$$
$$= \frac{\pi}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{\pi}{2} + \sqrt{3}$$

8 Maximum when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$6\cos 2x + 6\cos x = 0$$
$$6\left(2\cos^2 x - 1\right) + 6\cos x = 0$$
$$12\cos^2 x + 6\cos x - 6 = 0$$
$$2\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x - 1)(\cos x + 1) = 0$$

$\cos x = \frac{1}{2}$ or $\cos x = -1$

$x = \frac{\pi}{3}$ $\quad$ $x = \pi$

From the diagram, M occurs at $x = \frac{\pi}{3}$.

Area of shaded region $= \int_0^{\frac{\pi}{3}} (3\sin 2x + 6\sin x)\,\mathrm{d}x$

$$= \left[-\frac{3}{2}\cos 2x - 6\cos x\right]_0^{\frac{\pi}{3}}$$

$$= -\frac{3}{2}\cos\left(\frac{2\pi}{3}\right) - 6\cos\left(\frac{\pi}{3}\right) - \left(-\frac{3}{2}\cos 0 - 6\cos 0\right)$$

$$= -\frac{3}{2}\left(-\frac{1}{2}\right) - 6\left(\frac{1}{2}\right) + \frac{3}{2} + 6$$

$$= \frac{3}{4} - 3 + \frac{3}{2} + 6$$

$$= \frac{9}{4} + 3$$

$$= \frac{21}{4}$$

9 **a** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x \, dx = [-\cos x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

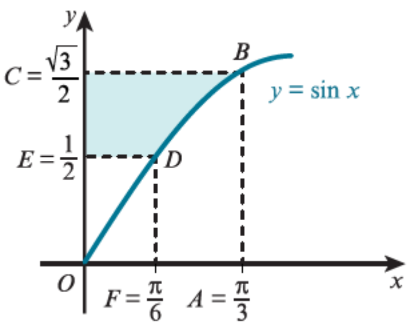
$$= -\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right)$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}$$

$$= \frac{1}{2}(\sqrt{3} - 1)$$

b

It is essential that you draw a diagram for this kind of question. Make it clear in your mind which area you are trying to find and then work out which other areas can be used to make the correct calculation.



This integral represents the shaded region in the diagram:

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} (\sin^{-1} y) \, dy$$

$$= \text{Area of rectangle } OABC - \text{Area of rectangle } OFDE - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x \, dx$$

$$= \frac{\pi}{3}\frac{\sqrt{3}}{2} - \frac{1}{2}\frac{\pi}{6} - \frac{1}{2}(\sqrt{3} - 1)$$

$$= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{12} - \frac{1}{2}\sqrt{3} + \frac{1}{2}$$

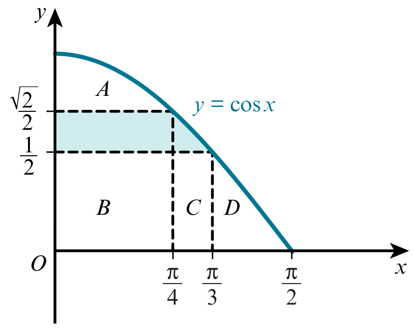
$$= \frac{\pi}{12}(2\sqrt{3} - 1) - \frac{\sqrt{3} - 1}{2}$$

10 **a** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos x \, dx = [\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$$= \sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} = \frac{\sqrt{3} - \sqrt{2}}{2}$$

b



$$\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \left(\cos^{-1} y\right) \mathrm{d}y = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \cos x \, \mathrm{d}x - C + \text{shaded rectangle}$$

$$= \frac{\sqrt{3}-\sqrt{2}}{2} - \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + \frac{\pi}{4}\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right)$$

$$= \frac{\sqrt{3}-\sqrt{2}}{2} - \frac{1}{2}\left(\frac{\pi}{12}\right) + \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right) - \frac{\pi}{8}$$

$$= \frac{\pi}{24}\left(-1 + 3\sqrt{2} - 3\right) + \frac{\sqrt{3}-\sqrt{2}}{2}$$

$$= \frac{\pi}{24}\left(3\sqrt{2} - 4\right) + \frac{\sqrt{3}-\sqrt{2}}{2}$$

EXERCISE 5D

1 **b** Although the coursebook shows the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, in part **b** of Question **1** you know that you will need to expand the brackets, so just use the expanded version from the very start.

$$\int 4\cos^2\left(\frac{x}{2}\right) \mathrm{d}x$$

$$4\int \left(\frac{1}{2} + \frac{1}{2}\cos x\right) \mathrm{d}x$$

$$= 4\left(\frac{1}{2}x + \frac{1}{2}\sin x\right) + c$$

$$= 2x + 2\sin x + c$$

e

$$\int 6\tan^2(3x)\,\mathrm{d}x = 6\int \left(\sec^2(3x) - 1\right) \mathrm{d}x$$

$$= \frac{6}{3}\tan 3x - 6x + c$$

$$= 2\tan 3x - 6x + c$$

2 **a**

$$\int_0^{\frac{\pi}{3}} \sin^2 x\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) \mathrm{d}x$$

$$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{6} - \frac{1}{4}\sin\left(\frac{2\pi}{3}\right) - 0 - 0$$

$$= \frac{\pi}{6} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{24}(4\pi - 3\sqrt{3})$$

f Rearranging $1 + \tan^2\theta \equiv \sec^2\theta$ gives $\tan^2\theta \equiv \sec^2\theta - 1$.

$$\int_0^{\frac{\pi}{6}} \tan^2 2x\,\mathrm{d}x = \int_0^{\frac{\pi}{6}} \left(\sec^2 2x - 1\right) \mathrm{d}x$$

$$= \left[\frac{1}{2}\tan 2x - x\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{6} - 0 + 0$$

$$= \frac{1}{2}(\sqrt{3}) - \frac{\pi}{6}$$

$$= \frac{3\sqrt{3} - \pi}{6}$$

3 **a**

$$\int_0^{\frac{\pi}{6}} \left(\cos^2 x - \frac{1}{\cos^2 x}\right) \mathrm{d}x$$

$$= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} + \frac{1}{2}\cos 2x - \sec^2 x\right)\,\mathrm{d}x$$
$$= \left[\frac{1}{2}x + \frac{1}{4}\sin 2x - \tan x\right]_0^{\frac{\pi}{6}}$$
$$= \frac{\pi}{12} + \frac{1}{4}\sin\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{6}\right) - 0 - 0 + 0$$
$$= \frac{\pi}{12} + \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{3}$$
$$= \frac{1}{24}(2\pi + 3\sqrt{3} - 8\sqrt{3})$$
$$= \frac{2\pi - 5\sqrt{3}}{24}$$

e $\displaystyle\int_0^{\frac{\pi}{6}} \frac{1 + \cos^4 x}{\cos^2 x}\,\mathrm{d}x$

$$\int_0^{\frac{\pi}{6}} \left(\frac{1}{\cos^2 x} + \cos^2 x\right)\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{6}} \left(\sec^2 x + \frac{1}{2} + \frac{1}{2}\cos 2x\right)\,\mathrm{d}x$$
$$= \left[\tan x + \frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{6}}$$
$$= \tan\left(\frac{\pi}{6}\right) + \frac{\pi}{12} + \frac{1}{4}\sin\left(\frac{\pi}{3}\right) - 0 - 0 - 0$$
$$= \frac{\sqrt{3}}{3} + \frac{\pi}{12} + \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{1}{24}(8\sqrt{3} + 2\pi + 3\sqrt{3})$$
$$= \frac{1}{24}(11\sqrt{3} + 2\pi)$$

4 **a** $y = \cot x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos x}{\sin x}\right)$$
$$= \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$
$$= -\frac{(\sin^2 x + \cos^2 x)}{\sin^2 x}$$
$$= -\frac{1}{\sin^2 x}$$
$$= -\operatorname{cosec}^2 x$$

b $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^2 2x\,\mathrm{d}x = \left[-\frac{1}{2}\cot 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$$= -\frac{1}{2}\cot\left(\frac{\pi}{2}\right) + \frac{1}{2}\cot\left(\frac{\pi}{3}\right)$$
$$= -0 + \frac{1}{2\tan\left(\frac{\pi}{3}\right)}$$
$$= \frac{1}{2\sqrt{3}}$$
$$= \frac{\sqrt{3}}{6}$$

5 **a** $\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$

$$= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$$

$$= \frac{1}{\sin x \cos x}$$

$$= \frac{2}{2 \sin x \cos x}$$

$$= \frac{2}{\sin 2x}$$

b

Always remember that the parts of questions might be linked. When integrating trigonometric functions, you are often asked to prove an identity which you will use in the second part of the question.

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{8}{\tan x + \cot x}\,dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{8}{\left(\frac{2}{\sin 2x}\right)}\,dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 4 \sin 2x\,dx$$

$$= [-2 \cos 2x]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= -2 \cos \pi + 2 \cos\left(\frac{2\pi}{3}\right)$$

$$= 2 - 2\left(\frac{1}{2}\right)$$

$$= 2 - 1$$

$$= 1$$

6 **a** $\sin 7x + \sin 3x$

$= \sin(5x + 2x) + \sin(5x - 2x)$

$= \sin 5x \cos 2x + \cos 5x \sin 2x + \sin 5x \cos 2x - \cos 5x \sin 2x$

$= 2 \sin 5x \cos 2x$

b $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (2 \sin 5x \cos 2x)\,dx$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sin 7x + \sin 3x)\,dx$$

$$= \left[-\frac{1}{7}\cos 7x - \frac{1}{3}\cos 3x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= -\frac{1}{7}\cos\left(\frac{7\pi}{3}\right) - \frac{1}{3}\cos \pi + \frac{1}{7}\cos\left(\frac{7\pi}{6}\right) + \frac{1}{3}\cos\left(\frac{\pi}{2}\right)$$

$$= -\frac{1}{7}\cos\left(\frac{\pi}{3}\right) + \frac{1}{3} + \frac{1}{7}\left(-\frac{\sqrt{3}}{2}\right) + 0$$

$$= -\frac{1}{7}\left(\frac{1}{2}\right) + \frac{1}{3} - \frac{\sqrt{3}}{14}$$

$$= -\frac{1}{14} + \frac{1}{3} - \frac{\sqrt{3}}{14}$$

$$= -\frac{3}{42} + \frac{14}{42} - \frac{3\sqrt{3}}{42}$$

$$= \frac{11 - 3\sqrt{3}}{42}$$

7 **a** $\sin 3x = \sin(2x + x)$

$$= \sin 2x \cos x + \cos 2x \sin x$$
$$= 2 \sin x \cos x \cos x + \left(1 - 2\sin^2 x\right) \sin x$$
$$= 2 \sin x \cos^2 x + \sin x - 2 \sin^3 x$$
$$= 2 \sin x(1 - \sin^2 x) + \sin x - 2 \sin^3 x$$
$$= 3 \sin x - 4 \sin^3 x$$

b Using the result from part **a**:

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$
$$4 \sin^3 x = 3 \sin x - \sin 3x$$
$$2 \sin^3 x = \frac{3}{2}\sin x - \frac{1}{2}\sin 3x$$

$$\int_0^{\frac{\pi}{3}} 2 \sin^3 x \, dx$$
$$= \int_0^{\frac{\pi}{3}} \left(\frac{3}{2}\sin x - \frac{1}{2}\sin 3x\right) dx$$
$$= \left[-\frac{3}{2}\cos x + \frac{1}{6}\cos 3x\right]_0^{\frac{\pi}{3}}$$
$$= -\frac{3}{2}\cos\left(\frac{\pi}{3}\right) + \frac{1}{6}\cos \pi + \frac{3}{2}\cos 0 - \frac{1}{6}\cos 0$$
$$= -\frac{3}{2}\left(\frac{1}{2}\right) - \frac{1}{6} + \frac{3}{2} - \frac{1}{6}$$
$$= -\frac{3}{4} - \frac{1}{6} + \frac{3}{2} - \frac{1}{6}$$
$$= \frac{-9 - 2 + 18 - 2}{12}$$
$$= \frac{5}{12}$$

8 **a** $\cos 3x = \cos(2x + x)$

$$= \cos 2x \cos x - \sin 2x \sin x$$
$$= \left(2 \cos^2 x - 1\right) \cos x - 2 \sin x \cos x \sin x$$
$$= 2 \cos^3 x - \cos x - 2 \sin^2 x \cos x$$
$$= 2 \cos^3 x - \cos x - 2(1 - \cos^2 x) \cos x$$
$$= 2 \cos^3 x - \cos x - 2 \cos x + 2 \cos^3 x$$
$$= 4 \cos^3 x - 3 \cos x$$

b Using the result from part **a**:

$$\cos 3x = 4 \cos^3 x - 3 \cos x$$
$$4 \cos^3 x = \cos 3x + 3 \cos x$$
$$4 \cos^3 x + 2 \cos x = \cos 3x + 5 \cos x$$

$$= \int_0^{\frac{\pi}{6}} \left(4 \cos^3 x + 2 \cos x\right) dx$$
$$= \int_0^{\frac{\pi}{6}} (\cos 3x + 5 \cos x) \, dx$$
$$= \left[\frac{1}{3}\sin 3x + 5 \sin x\right]_0^{\frac{\pi}{6}}$$
$$= \frac{1}{3}\sin\left(\frac{\pi}{2}\right) + 5 \sin\left(\frac{\pi}{6}\right) - 0 - 0$$
$$= \frac{1}{3}(1) + 5\left(\frac{1}{2}\right)$$
$$= \frac{5}{2} + \frac{1}{3}$$
$$= \frac{15 + 2}{6}$$
$$= \frac{17}{6}$$

9 $V = \int_0^{\frac{\pi}{2}} \pi y^2 \, dx$

$$= \int_0^{\frac{\pi}{2}} \pi(\cos x + 2\sin x)^2 \, dx$$

$$= \int_0^{\frac{\pi}{2}} \pi \left(\cos^2 x + 4\sin x \cos x + 4\sin^2 x\right) \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2x + 2\sin 2x + \frac{4}{2} - \frac{4}{2}\cos 2x\right) dx$$

$$= \pi \left[\frac{5}{2}x + \frac{1}{4}\sin 2x - \cos 2x - \sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \pi \left[\frac{5}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin \pi - \cos \pi - \sin \pi - 0 - 0 + 1 + 0\right]$$

$$= \frac{5\pi^2}{4} + 0 + \pi - 0 + \pi$$

$$= \frac{5\pi^2}{4} + 2\pi$$

$$= \pi\left(2 + \frac{5\pi}{4}\right)$$

10

When proving identities, always look carefully at the form of the final expression that you are trying to get to. In part **a** of Question **10**, the original expression contains terms in $\sec x$ and $\tan x$, but the final expression contains only $\sin x$. How can you convert to terms in $\sin x$ only?

a $\sec^2 x + \sec x \tan x \equiv \frac{1}{\cos^2 x} + \frac{1}{\cos x}\frac{\sin x}{\cos x}$

$$\equiv \frac{1 + \sin x}{\cos^2 x}$$

$$\equiv \frac{1 + \sin x}{1 - \sin^2 x}$$

$$\equiv \frac{1 + \sin x}{(1 - \sin x)(1 + \sin x)}$$

$$\equiv \frac{1}{1 - \sin x}$$

b $y = \sec x = (\cos x)^{-1}$

$$\frac{dy}{dx} = -(\cos x)^{-2}(-\sin x)$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$$

$$= \sec x \tan x$$

c $\int_0^{\frac{\pi}{6}} \frac{4}{1 - \sin 2x} \, dx$

$$= 4\int_0^{\frac{\pi}{6}} \frac{1}{1 - \sin 2x} \, dx$$

$$= 4\int_0^{\frac{\pi}{6}} \left(\sec^2 2x + \sec 2x \tan 2x\right) dx \text{ (from the result in part } \mathbf{a}\text{)}$$

$$= 4\left[\frac{1}{2}\tan 2x + \frac{1}{2}\sec 2x\right]_0^{\frac{\pi}{6}} \text{ (from the result in part } \mathbf{b}\text{)}$$

$$= 4\left(\frac{1}{2}\tan\frac{\pi}{3} + \frac{1}{2}\sec\frac{\pi}{3} - 0 - \frac{1}{2}\sec 0\right)$$
$$= 2\sqrt{3} + \frac{2}{\cos\frac{\pi}{3}} - \frac{4}{2}$$
$$= 2\sqrt{3} + \frac{2}{\left(\frac{1}{2}\right)} - 2$$
$$= 2\sqrt{3} + 4 - 2$$
$$= 2\sqrt{3} + 2$$
$$= 2(1 + \sqrt{3})$$

P2 EXERCISE 5E

1 **a** $a = 2$

$b = 4$

$h = \dfrac{4 - 2}{2} = 1$

x	2	3	4
y	$\sqrt{2}$	$\sqrt{7}$	$\sqrt{14}$
	y_0	y_1	y_2

$$\int_2^4 \sqrt{x^2 - 2}\,dx \approx \frac{h}{2}[y_0 + y_2 + 2(y_1)]$$

$$= \frac{1}{2}[\sqrt{2} + \sqrt{14} + 2\sqrt{7}]$$

$$= 5.22 \text{ (to 2 decimal places)}$$

> Try to avoid working out each y-value as a decimal. It is best to keep exact values until the very end.

f $a = 2$

$b = 12$

$h = \dfrac{12 - 2}{2} = 5$

x	2	7	12
y	$\log_{10} 2$	$\log_{10} 7$	$\log_{10} 12$
	y_0	y_1	y_2

$$\int_2^{12} \log_{10} x\,dx \approx \frac{h}{2}[y_0 + y_2 + 2(y_1)]$$

$$= \frac{5}{2}[\log_{10} 2 + \log_{10} 12 + 2\log_{10} 7]$$

$$= 7.68 \text{ (to 2 decimal places)}$$

2 $a = 1$

$b = 5$

$h = \dfrac{5 - 1}{4} = 1$

x	1	2	3	4	5
y	e^{-1}	$4e^{-2}$	$9e^{-3}$	$16e^{-4}$	$25e^{-5}$
	y_0	y_1	y_2	y_3	y_4

$$\int_1^5 x^2 e^{-x}\,dx \approx \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1}{2}[e^{-1} + 25e^{-5} + 2(4e^{-2} + 9e^{-3} + 16e^{-4})]$$

$$= 1.55 \text{ (to 2 decimal places)}$$

3 **a** $a = \dfrac{\pi}{4}$

$b = \dfrac{7\pi}{4}$

$h = \dfrac{\frac{7\pi}{4} - \frac{\pi}{4}}{6} = \dfrac{\pi}{4}$

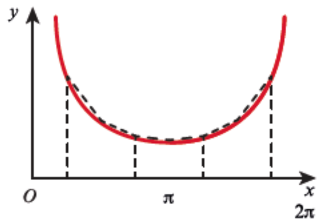
x	$\frac{\pi}{4}$	$\frac{2\pi}{4}$	$\frac{3\pi}{4}$	$\frac{4\pi}{4}$	$\frac{5\pi}{4}$	$\frac{6\pi}{4}$	$\frac{7\pi}{4}$
y	$\operatorname{cosec}\left(\frac{\pi}{8}\right)$	$\operatorname{cosec}\left(\frac{2\pi}{8}\right)$	$\operatorname{cosec}\left(\frac{3\pi}{8}\right)$	$\operatorname{cosec}\left(\frac{4\pi}{8}\right)$	$\operatorname{cosec}\left(\frac{5\pi}{8}\right)$	$\operatorname{cosec}\left(\frac{6\pi}{8}\right)$	$\operatorname{cosec}\left(\frac{7\pi}{8}\right)$

$$\int_{\frac{\pi}{4}}^{\frac{7\pi}{4}} \operatorname{cosec}\left(\frac{1}{2}x\right) \mathrm{d}x \approx \frac{h}{2}[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{8}\left[\operatorname{cosec}\left(\frac{\pi}{8}\right) + \operatorname{cosec}\left(\frac{7\pi}{8}\right) + 2\left\{\operatorname{cosec}\left(\frac{2\pi}{8}\right) + \operatorname{cosec}\left(\frac{3\pi}{8}\right) + \operatorname{cosec}\left(\frac{4\pi}{8}\right) + \operatorname{cosec}\left(\frac{5\pi}{8}\right) + \operatorname{cosec}\left(\frac{6\pi}{8}\right)\right\}\right]$$

$$= 6.76$$

b This is an over-estimate since the top edges of the strips all lie above the curve.



4 $a = 0$

$b = \frac{\pi}{2}$

$h = \frac{\frac{\pi}{2} - 0}{3} = \frac{\pi}{6}$

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$
y	$e^0 \cos 0$	$e^{\frac{\pi}{6}} \cos \frac{\pi}{6}$	$e^{\frac{2\pi}{6}} \cos \frac{2\pi}{6}$	$e^{\frac{3\pi}{6}} \cos \frac{3\pi}{6}$

$$\int_0^{\frac{\pi}{2}} e^x \cos x \,\mathrm{d}x \approx \frac{h}{2}[y_0 + y_3 + 2(y_1 + y_2)]$$

$$= \frac{\pi}{12}\left[e^0 \cos 0 + e^{\frac{3\pi}{6}} \cos \frac{3\pi}{6} + 2\left\{e^{\frac{\pi}{6}} \cos \frac{\pi}{6} + e^{\frac{2\pi}{6}} \cos \frac{2\pi}{6}\right\}\right]$$

$$= 1.77 \text{ (to 2 decimal places)}$$

This is an under-estimate because the curve is convex and the top edges of the strips all lie below the curve.

5 $a = 1$

$b = 3$

$h = \frac{3 - 1}{4} = \frac{1}{2}$

x	1	1.5	2	2.5	3
y	$\frac{e^1}{2(1)}$	$\frac{e^{1.5}}{2(1.5)}$	$\frac{e^2}{2(2)}$	$\frac{e^{2.5}}{2(2.5)}$	$\frac{e^3}{2(3)}$

$$\int_1^3 \frac{e^x}{2x} \mathrm{d}x \approx \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1}{4}\left[\frac{e^1}{2(1)} + \frac{e^3}{2(3)} + 2\left\{\frac{e^{1.5}}{2(1.5)} + \frac{e^2}{2(2)} + \frac{e^{2.5}}{2(2.5)}\right\}\right]$$

$$= 4.07 \text{ (to 2 decimal places)}$$

This is an over-estimate, because the curve is concave and the top edges of the strips all lie above the curve.

END-OF-CHAPTER REVIEW EXERCISE 5

1 $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 + 2\tan^2 x)\, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 + 2(\sec^2 x - 1))\, dx$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (2\sec^2 x - 1)\, dx$$
$$= [2\tan x - x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= 2\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3} - \left(2\tan\left(\frac{\pi}{6}\right) - \frac{\pi}{6}\right)$$
$$= 2\sqrt{3} - \frac{\pi}{3} - 2\left(\frac{\sqrt{3}}{3}\right) + \frac{\pi}{6}$$
$$= 2\sqrt{3} - \frac{2}{3}\sqrt{3} - \frac{\pi}{6}$$
$$= \frac{4}{3}\sqrt{3} - \frac{\pi}{6}$$

2 $$\int_{-4}^{-2} \left(\frac{4}{1-2x} - 2x\right) dx = -2\int_{-4}^{-2} \left(\frac{-2}{1-2x} + x\right) dx$$
$$= -2\left[\ln|1-2x| + \frac{1}{2}x^2\right]_{-4}^{-2}$$
$$= -2(\ln 5 + 2 - \ln 9 - 8)$$
$$= -2\ln\left(\frac{5}{9}\right) + 12$$
$$= 12 + 2\ln\left(\frac{9}{5}\right)$$
$$= 12 + \ln\left(\frac{81}{25}\right)$$

P2

3 $a = 0$

$b = 1$

$h = \frac{1-0}{4} = \frac{1}{4}$

x	0	0.25	0.5	0.75	1
y	$2 - 0^2\ln(0+1)$	$2 - 0.25^2\ln(0.25+1)$	$2 - 0.5^2\ln(0.5+1)$	$2 - 0.75^2\ln(0.75+1)$	$2 - 1^2\ln(1+1)$

$$\int_0^1 (2 - x^2\ln(x+1))\, dx \approx \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$
$$= \frac{1}{8}[2 - 0^2\ln(0+1) + 2 - 1^2\ln(1+1) + 2\{2 - 0.25^2\ln(0.25+1)$$
$$= +2 - 0.5^2\ln(0.5+1) + 2 - 0.75^2\ln(0.75+1)\}]$$
$= 1.81$ (to 2 decimal places)

This is an under-estimate because the top edges of the strips all lie below the curve.

4 $$\int_0^{\frac{\pi}{6}} (\cos 3x - \sin 2x)\, dx = \left[\frac{1}{3}\sin 3x + \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}}$$
$$= \frac{1}{3}\sin\frac{\pi}{2} + \frac{1}{2}\cos\frac{\pi}{3} - 0 - \frac{1}{2}\cos 0$$
$$= \frac{1}{3} + \frac{1}{2}\left(\frac{1}{2}\right) - 0 - \frac{1}{2}$$
$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{2}$$
$$= \frac{4+3-6}{12}$$
$$= \frac{1}{12}$$

5 $$\int_0^1 (3e^x - 2)^2\,dx = \int_0^1 \left(9e^{2x} - 12e^x + 4\right)dx$$
$$= \left[\frac{9}{2}e^{2x} - 12e^x + 4x\right]_0^1$$
$$= \frac{9}{2}e^2 - 12e^1 + 4 - \frac{9}{2}e^0 + 12e^0 - 0$$
$$= \frac{9}{2}e^2 - 12e + 4 - \frac{9}{2} + 12$$
$$= \frac{9}{2}e^2 - 12e + \frac{23}{2}$$

6 **a** $$\int_0^k (5e^{-2x} + 2e^{-3x})\,dx = \left[-\frac{5}{2}e^{-2x} - \frac{2}{3}e^{-3x}\right]_0^k$$
$$= -\frac{5}{2}e^{-2k} - \frac{2}{3}e^{-3k} - \left(-\frac{5}{2}e^0 - \frac{2}{3}e^0\right)$$
$$= -\frac{5}{2}e^{-2k} - \frac{2}{3}e^{-3k} + \frac{5}{2} + \frac{2}{3}$$
$$= -\frac{5}{2}e^{-2k} - \frac{2}{3}e^{-3k} + \frac{19}{6}$$

b $$\int_0^\infty (5e^{-2x} + 2e^{-3x})\,dx = \lim_{k\to\infty}\left(-\frac{5}{2}e^{-2k} - \frac{2}{3}e^{-3k} + \frac{19}{6}\right)$$
$$= 0 + 0 + \frac{19}{6}$$
$$= \frac{19}{6}$$

7 $$\int_2^4 \frac{3}{5x+1}\,dx = \frac{3}{5}\int_2^4 \frac{5}{5x+1}\,dx$$
$$= \frac{3}{5}[\ln|5x+1|]_2^4$$
$$= \frac{3}{5}\ln 21 - \frac{3}{5}\ln 11$$
$$= \frac{3}{5}\ln\left(\frac{21}{11}\right)$$

8 **a** $$4 + \frac{A}{2x+5} \equiv \frac{8x + 20 + A}{2x+5} \equiv \frac{8x}{2x+5}$$
$$20 + A = 0$$
$$A = -20$$

b Using the result from part **a**:

$$\int_1^3 \frac{8x}{2x+5}\,dx = \int_1^3 \left(4 - \frac{20}{2x+5}\right)dx$$
$$= [4x - 10\ln|2x+5|]_1^3$$
$$= 12 - 10\ln 11 - (4 - 10\ln 7)$$
$$= 12 - 10\ln 11 - 4 + 10\ln 7$$
$$= 8 - 10\ln\left(\frac{11}{7}\right)$$

9 **i** $$12\sin^2 x\cos^2 x \equiv 3\left(4\sin^2 x\cos^2 x\right)$$
$$\equiv 3(2\sin x\cos x)^2$$
$$\equiv 3\sin^2 2x$$

But $\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$

so $12\sin^2 x\cos^2 x = 3\left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) = \frac{3}{2}(1 - \cos 4x)$

ii Using the result from part **i**:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 12\sin^2 x\cos^2\ \mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{3}{2} - \frac{3}{2}\cos 4x\right)\mathrm{d}x$$
$$= \left[\frac{3}{2}x + \frac{3}{8}\sin 4x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$= \frac{3\pi}{6} + \frac{3}{8}\sin\frac{4\pi}{3} - \left(\frac{3\pi}{8} - \frac{3}{8}\sin\pi\right)$$
$$= \frac{\pi}{2} + \frac{3}{8}\left(\frac{\sqrt{3}}{2}\right) - \frac{3\pi}{8} + 0$$
$$= \frac{\pi}{8} + \frac{3\sqrt{3}}{16}$$

10 **a** $\int 4\mathrm{e}^x(3 + \mathrm{e}^{2x})\,\mathrm{d}x = \int (12\mathrm{e}^x + 4\mathrm{e}^{3x})\,\mathrm{d}x$

$$= 12\mathrm{e}^x + \frac{4}{3}\mathrm{e}^{3x} + c$$

b $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(3 + 2\tan^2\theta)\,\mathrm{d}\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}[3 + 2(\sec^2\theta - 1)]\,\mathrm{d}\theta$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(3 + 2\sec^2\theta - 2)\,\mathrm{d}\theta$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1 + 2\sec^2\theta)\,\mathrm{d}\theta$$
$$= [\theta + 2\tan\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$
$$= \frac{\pi}{4} + 2\tan\frac{\pi}{4} - \left(-\frac{\pi}{4} + 2\tan\left(-\frac{\pi}{4}\right)\right)$$
$$= \frac{\pi}{2} + 2 - (-2)$$
$$= \frac{\pi}{2} + 4$$
$$= \frac{1}{2}(\pi + 8)$$

11 **i** $(2\sin x + \cos x)^2$

$= 4\sin^2 x + 4\sin x\cos x + \cos^2 x$

$= 3\sin^2 x + 2(2\sin x\cos x) + \sin^2 x + \cos^2 x$

$= 3\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) + 2\sin 2x + 1$

$= \frac{3}{2} + 1 - \frac{3}{2}\cos 2x + 2\sin 2x$

$= \frac{5}{2} + 2\sin 2x - \frac{3}{2}\cos 2x$

ii Using the result from part **i**:

$\int_0^{\frac{\pi}{4}}(2\sin x + \cos x)^2\,\mathrm{d}x$

$\int_0^{\frac{\pi}{4}}\left(\frac{5}{2} + 2\sin 2x - \frac{3}{2}\cos 2x\right)\mathrm{d}x$

$= \left[\frac{5}{2}x - \cos 2x - \frac{3}{4}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \frac{5}{2}\left(\frac{\pi}{4}\right) - \cos\frac{\pi}{2} - \frac{3}{4}\sin\frac{\pi}{2} - 0 + 1 - 0$

$= \frac{5\pi}{8} - \frac{3}{4} + 1$

$= \frac{5\pi}{8} + \frac{1}{4}$

$= \frac{1}{8}(5\pi + 2)$

12 **i** $y = \cot x = \dfrac{\cos x}{\sin x}$

$$\frac{dy}{dx} = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$
$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$
$$= -\frac{1}{\sin^2 x}$$
$$= -\operatorname{cosec}^2 x$$

ii $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$

$\cot^2 x \equiv \operatorname{cosec}^2 x - 1$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^2 x \, dx$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\operatorname{cosec}^2 x - 1) \, dx$$
$$= [-\cot x - x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \text{ (from the result in part } \mathbf{i}\text{)}$$
$$= -\cot \frac{\pi}{2} - \frac{\pi}{2} - \left(-\frac{1}{\tan \frac{\pi}{4}} - \frac{\pi}{4}\right)$$
$$= 0 - \frac{\pi}{2} + 1 + \frac{\pi}{4}$$
$$= 1 - \frac{\pi}{4}$$

iii $\cos 2x = 1 - 2\sin^2 x$

$$\frac{1}{1 - \cos 2x}$$
$$= \frac{1}{1 - (1 - 2\sin^2 x)}$$
$$= \frac{1}{2\sin^2 x}$$
$$= \frac{1}{2}\operatorname{cosec}^2 x$$

$$\int \frac{1}{1 - \cos 2x} \, dx$$
$$= \int \frac{1}{2}\operatorname{cosec}^2 x \, dx$$
$$= -\frac{1}{2}\cot x + c \text{ (from part } \mathbf{i}\text{)}$$

13 **i**
$$\int \frac{1 + \cos^4 2x}{\cos^2 2x} \, dx = \int \left(\frac{1}{\cos^2 2x} + \cos^2 2x\right) dx$$
$$= \int \left(\sec^2 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$
$$= \frac{1}{2}\tan 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x + c$$

ii
$$\int_4^{14} \left(2 + \frac{6}{3x - 2}\right) dx = [2x + 2\ln|3x - 2|]_4^{14}$$
$$= 28 + 2\ln 40 - (8 + 2\ln 10)$$
$$= 20 + 2\ln 40 - 2\ln 10$$
$$= 20 + 2\ln \frac{40}{10}$$
$$= 20 + 2\ln 4$$
$$= \ln e^{20} + \ln 4^2$$
$$= \ln(16e^{20})$$

14 **i** $y = x\sin x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = x\cos x + \sin x$

At the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) = 1$

Gradient of the normal is $-\frac{1}{1} = -1$

Equation of the normal is:

$$y - \frac{\pi}{2} = -1\left(x - \frac{\pi}{2}\right)$$
$$y = -x + \pi$$

When $x = \pi, y = -\pi + \pi = 0$

The normal passes through the point $(\pi, 0)$.

ii $$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x - x\cos x) = \cos x - (-x\sin x + \cos x)$$
$$= x\sin x$$

iii Using the result from part **ii**:

$$\int_0^{\frac{\pi}{2}} x\sin x\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}}{\mathrm{d}x}(\sin x - x\cos x)\,\mathrm{d}x$$
$$= [\sin x - x\cos x]_0^{\frac{\pi}{2}}$$
$$= \sin\frac{\pi}{2} - \frac{\pi}{2}\cos\frac{\pi}{2} - (\sin 0 + 0)$$
$$= 1 - 0 - 0 - 0$$
$$= 1$$

15 **i** $$\frac{1}{2}(\cos 2x - \cos 4x) \equiv \frac{1}{2}[\cos(3x - x) - \cos(3x + x)]$$
$$\equiv \frac{1}{2}[\cos 3x\cos x + \sin 3x\sin x - (\cos 3x\cos x - \sin 3x\sin x)]$$
$$\equiv \frac{1}{2}[\cos 3x\cos x - \cos 3x\cos x + \sin 3x\sin x + \sin 3x\sin x]$$
$$= \frac{1}{2}(2\sin 3x\sin x)$$
$$= \sin 3x\sin x$$

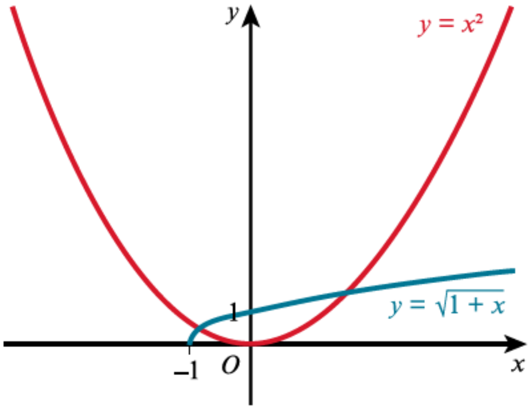
ii Using the result from part **i**:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 3x\sin x\,\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}(\cos 2x - \cos 4x)\,\mathrm{d}x$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x\right)\mathrm{d}x$$
$$= \left[\frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= \frac{1}{4}\sin\frac{2\pi}{3} - \frac{1}{8}\sin\frac{4\pi}{3} - \left(\frac{1}{4}\sin\frac{\pi}{3} - \frac{1}{8}\sin\frac{2\pi}{3}\right)$$
$$= \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{8}\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{8}\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{1}{8}\sqrt{3}$$

Chapter 6
Numerical solutions of equations

EXERCISE 6A

1 **a**



b The graphs intersect in two different places, so there are two roots.

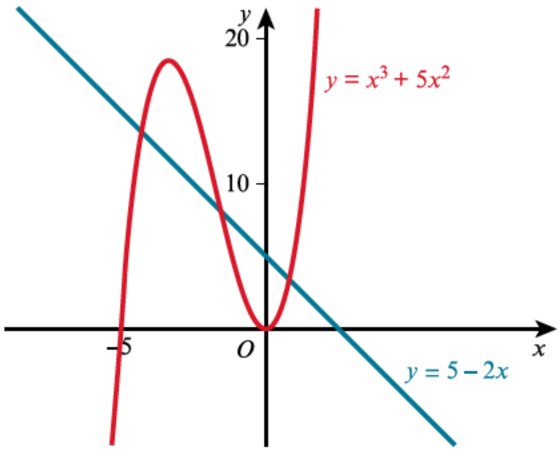
c Let $(x) = x^2 - \sqrt{1+x} = 0$, then

$f(-1) = (-1)^2 - \sqrt{1-1} = 1 > 0$ and

$f(0) = 0^2 - \sqrt{1+0} = -1 < 0$

Change of sign indicates that there is a root between -1 and 0.

2 **a**



Three points of intersection indicate that there are three real roots.

b Let $f(x) = x^3 + 5x^2 + 2x - 5 = 0$, then

$f(0) = 0 + 0 + 0 - 5 < 0$ and

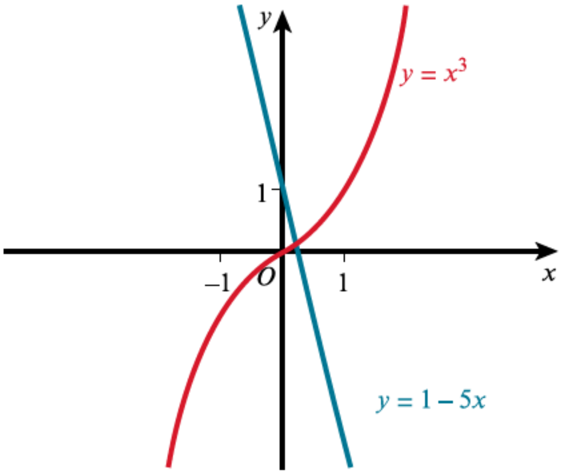
$f(2) = 8 + 20 + 4 - 5 > 0$

Change of sign indicates that there is a root between 0 and 2.

3 **a** You need to draw the graphs of $y = x^3$ and $y = 1 - 5x$ because they meet when

$x^3 = 1 - 5x$

i.e. when $x^3 + 5x - 1 = 0$.



There is only one point of intersection, so there is only one solution to $x^3 + 5x - 1 = 0$, and so only one point where the curve $y = x^3 + 5x - 1$ cuts the x-axis.

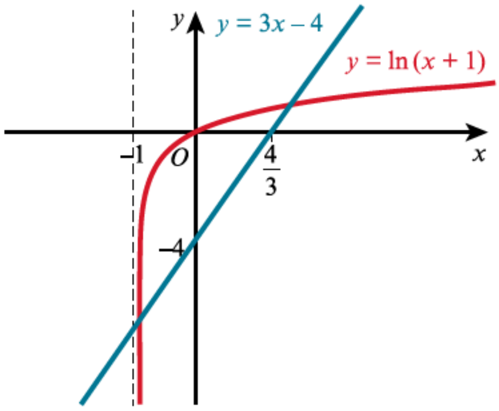
b Let $f(x) = x^3 + 5x - 1 = 0$, then

$f(0.1) = 0.001 + 0.5 - 1 < 0$ and

$f(0.5) = 0.0125 + 2.5 - 1 > 0$

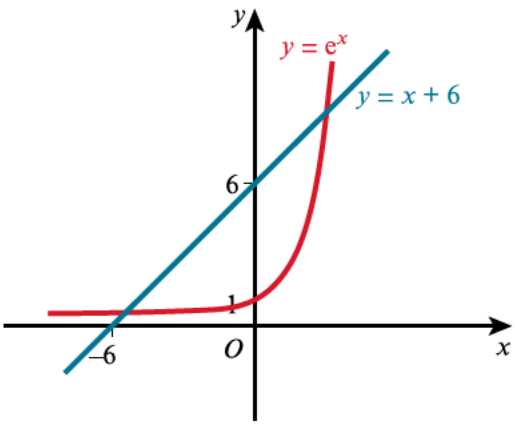
Change of sign indicates that there is a root between 0.1 and 0.5.

4 **a**



b The graphs intersect in two places, so there are two solutions.

5 **a**



There are two intersections, so there are two roots.

b Let $f(x) = e^x - x - 6 = 0$, then

$f(2.0) = e^2 - 2 - 6 = -0.6109 < 0$ and

$f(2.1) = e^{2.1} - 2.1 - 6 = 0.06617 > 0$

Change of sign indicates that there is a root between 2.0 and 2.1.

6

When using the change of sign method, you must remember to rearrange the original equation into the form $f(x) = 0$.

a Let $f(x) = (x + 2)e^{5x} - 1 = 0$, then

$f(0) = 2e^0 - 1 = 1 > 0$ and

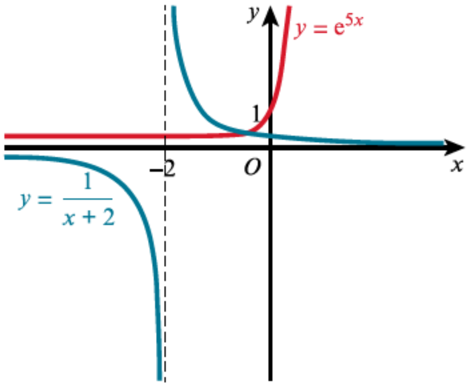
$f(-0.2) = 1.8e^{-1} - 1 = -0.33782 < 0$

Change of sign indicates that there is a root between 0 and -0.2.

b You need to draw the graphs of $y = e^{5x}$ and $y = \frac{1}{x+2}$ because if

$(x + 2)e^{5x} = 1$ then

$e^{5x} = \frac{1}{x+2}$.



Graphs only intersect at one point, so there is only one root.

7

Note that the question does not use any degree symbols. You must, therefore, put your calculator into radian mode.

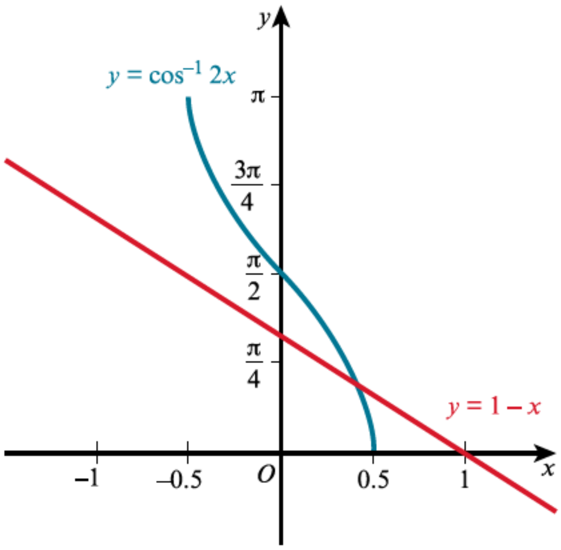
a Let $f(x) = \cos^{-1}2x + x - 1 = 0$, then

$f(0.4) = \cos^{-1}0.8 + 0.4 - 1 = 0.0435 > 0$ and

$f(0.5) = \cos^{-1}1 + 0.5 - 1 = -0.5 < 0$

Change of sign indicates that there is a root between 0.4 and 0.5.

b

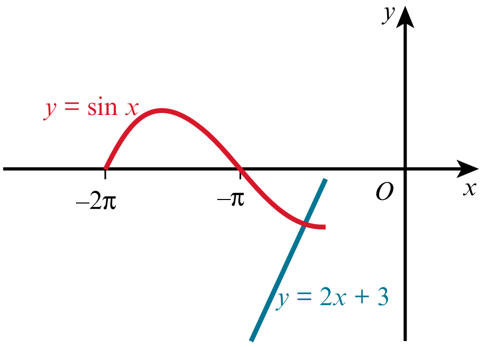


There is only one intersection in the domain $-0.5 \leqslant x \leqslant 0.5$, so there is only one root.

8 **a** $\frac{\sin x}{2x + 3} = 1$, so

$\sin x = 2x + 3$

You need to draw the graphs of $y = \sin x$ and $y = 2x + 3$.



There is only one point of intersection for $-2\pi < x < -\frac{\pi}{2}$, and so there is only one root of $1 = \frac{\sin x}{2x+3}$ on this domain. Also, if x is less than -2π or greater than $-\frac{\pi}{2}$, then the line and curve will not intersect again, so this is the only point of intersection of $y = \frac{\sin x}{2x+3}$ and $y = 1$.

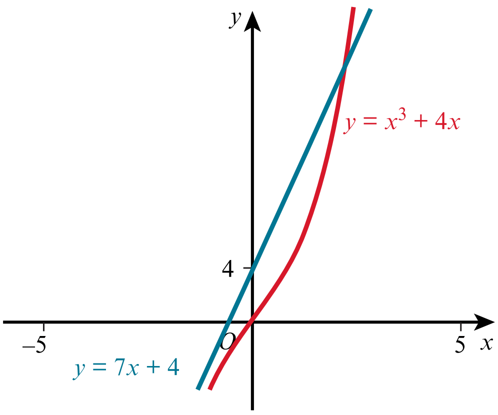
b Let $f(x) = \frac{\sin x}{2x+3} - 1 = 0$ then

$f(-2) = \frac{\sin(-2)}{-4+3} - 1 = -0.0907 < 0$ and

$f(-1.9) = \frac{\sin(-1.9)}{-3.8+3} - 1 = 0.1829 > 0$

Change of sign indicates that there is a root between -2 and -1.9, i.e. that $-2 < \alpha < -1.9$.

9 **a**



There is only one point of intersection for $0 \leqslant x \leqslant 5$, and so there is only one root on this domain.

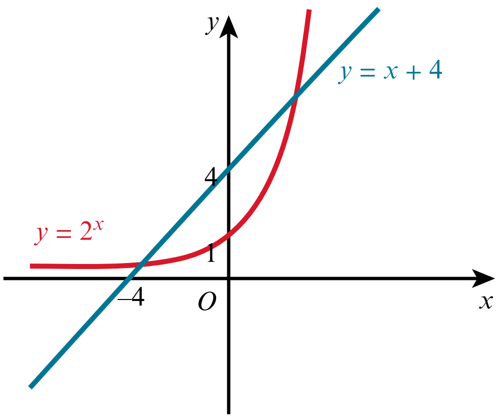
b Let $f(x) = x^3 - 3x - 4 = 0$ then

$f(2) = 8 - 6 - 4 = -2 < 0$ and

$f(3) = 27 - 9 - 4 = 14 > 0$

Change of sign indicates that the root lies between 2 and 3.

10 **a**



There are two intersections, so there are two roots.

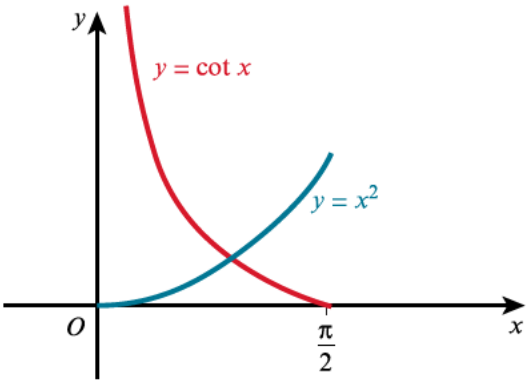
b Let $f(x) = 2^x - x - 4 = 0$ then

$f(2.7) = 2^{2.7} - 2.7 - 4 = -0.20198 < 0$ and

$f(2.8) = 2^{2.8} - 2.8 - 4 = 0.16440 > 0$

Change of sign indicates that there is a root between 2.7 and 2.8.

11 **a**



There is one point of intersection for $0 \leqslant x \leqslant \frac{\pi}{2}$, and so there is one root on this domain.

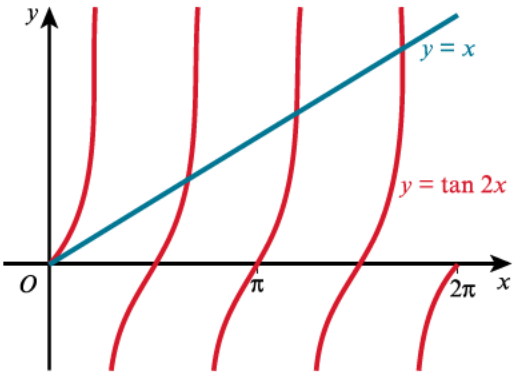
b Let $f(x) = \cot x - x^2 = 0$ then

$f(0.8) = \cot 0.8 - 0.8^2 = 0.33121 > 0$ and

$f(1) = \cot 1 - 1^2 = -0.35791 < 0$

Change of sign indicates that the root lies between 0.8 and 1.

12 **a**



There are three points of intersection for $0 < x < 2\pi$, and so there are three roots on this domain.

Pay careful attention to the inequality signs used in each question. In Question **12 a** the inequalities are strict, which actually excludes 0 as a solution. Always check this carefully.

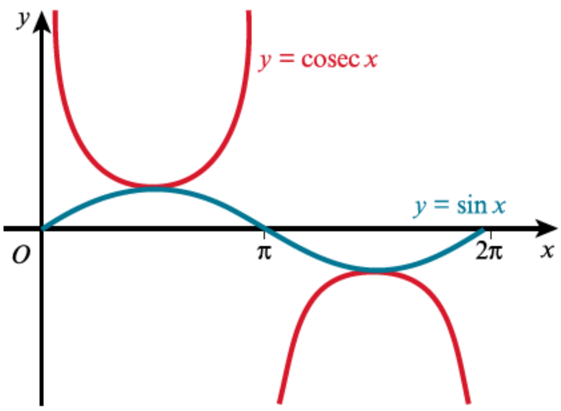
b Let $f(x) = x - \tan 2x = 0$ then

$f(2.1) = 2.1 - \tan 4.2 = 0.32222 > 0$ and

$f(2.2) = 2.2 - \tan 4.4 = -0.89632 < 0$

Change of sign indicates that there is a root between 2.1 and 2.2.

13 **a**



There are two points of intersection for $0 < x < 2\pi$, and so there are two roots on this domain.

b $\operatorname{cosec} x = \sin x$

$$\frac{1}{\sin x} = \sin x$$

$$\sin^2 x = 1$$

$$\sin x = \pm 1$$

$$x = \frac{\pi}{2} \text{ or } x = \frac{3\pi}{2}$$

The larger root is $x = \frac{3\pi}{2} = 4.71$ (to 3 s.f.).

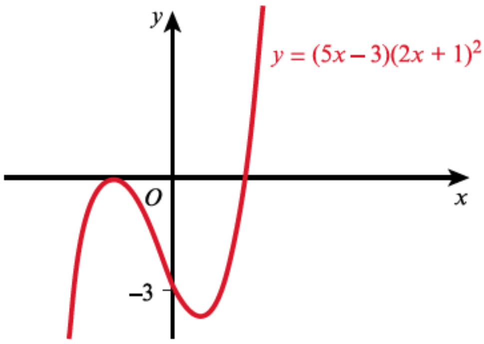
14 **a** $f(x) = 20x^3 + 8x^2 - 7x - 3$, so

$f(0.5) = 20\,(0.125) + 8\,(0.25) - 7\,(0.5) - 3 = -2 < 0$ and

$f(1) = 20\,(1) + 8\,(1) - 7\,(1) - 3 = 18 > 0$

Change of sign indicates that one of the roots is between 0.5 and 1.

b



Here is a table of values of $f(x)$ for various values of x. Plot the points and sketch the graph. You will then see that if there is a smaller root then it lies between -1 and 0. In fact, the graph only touches the x-axis here.

x	-1	-0.5	0	0.5	1
y	-8	0	-3	-2	18

c $$f\left(-\frac{1}{2}\right) = 20\left(-\frac{1}{2}\right)^3 + 8\left(-\frac{1}{2}\right)^2 - 7\left(-\frac{1}{2}\right) - 3$$

$$= -\frac{20}{8} + \frac{8}{4} + \frac{7}{2} - 3$$

$$= \frac{-20 + 16 + 28 - 24}{8}$$

$$= 0$$

By the factor theorem, $2x + 1$ is a factor of $f(x)$.

This means that you can factorise the cubic and, thus, solve it algebraically.

15 **a**

$$800 = \text{Volume (cylinder)} + \text{volume (hemisphere)}$$

$$800 = \pi r^2 \times 20 + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$$

$$800 = 20\pi r^2 + \frac{2}{3}\pi r^3$$

$$2400 = 60\pi r^2 + 2\pi r^3$$

$$\pi r^3 + 30\pi r^2 - 1200 = 0$$

b Let $f(r) = \pi r^3 + 30\pi r^2 - 1200 = 0$ then

$f(3) = 27\pi + 270\pi - 1200 = -266.947 < 0$ and

$f(4) = 64\pi + 480\pi - 1200 = 509.026 > 0$

Change of sign indicates that there is a root between 3 and 4.

EXERCISE 6B

1 **a** Using $x_{n+1} = \dfrac{7 - {x_n}^3}{5}$ with $x_1 = 1.1$:

$$x_2 = \frac{7 - 1.1^3}{5} = 1.1338$$

$$x_3 = \frac{7 - 1.1338^3}{5} = 1.1085$$

$$x_4 = \frac{7 - 1.1085^3}{5} = 1.1276$$

$$x_5 = \frac{7 - 1.1276^3}{5} = 1.1133$$

$$x_6 = \frac{7 - 1.1133^3}{5} = 1.1240$$

b $x_6 = 1.12$ (to 2 decimal points)

$\mathrm{f}(1.115) = 1.115^3 + (5 \times 1.115) - 7 = -0.0388$

$\mathrm{f}(1.125) = 1.125^3 + (5 \times 1.125) - 7 = 0.0488$

Change of sign indicates that there is a root between 1.115 and 1.125.

The root therefore has the value 1.12 to 2 decimal places.

2 **a** Let $\mathrm{f}(x) = \ln\ (x + 1) + 2x - 4 = 0$ then

$\mathrm{f}(1) = \ln(2) + 2 - 4 = -1.307$ and

$\mathrm{f}(2) = \ln(3) + 4 - 4 = 1.099$

Change of sign indicates that there is a root between 1 and 2.

b Using $x_{n+1} = \dfrac{4 - \ln\ (x_n + 1)}{2}$ with $x_1 = 1.5$:

$$x_2 = \frac{4 - \ln(2.5)}{2} = 1.541855$$

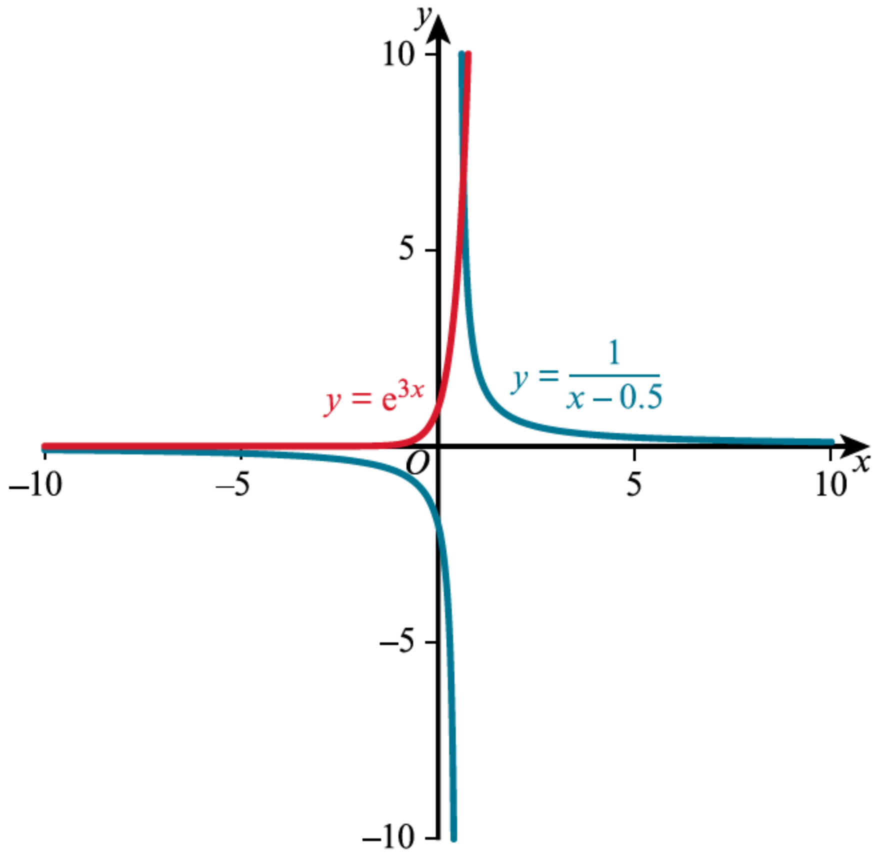
$$x_3 = \frac{4 - \ln(2.541855)}{2} = 1.533553$$

$$x_4 = \frac{4 - \ln(2.533553)}{2} = 1.535189$$

$$x_5 = \frac{4 - \ln(2.535189)}{2} = 1.534866$$

So $\alpha = 1.535$ (to 3 decimal points).

3 **a** $e^{3x} = \dfrac{1}{x - 0.5}$, so drawing the graphs of $y = e^{3x}$ and $y = \dfrac{1}{x - 0.5}$



There is only one point of intersection, so there is only one root of the equation $(x - 0.5)\,e^{3x} = 1$.

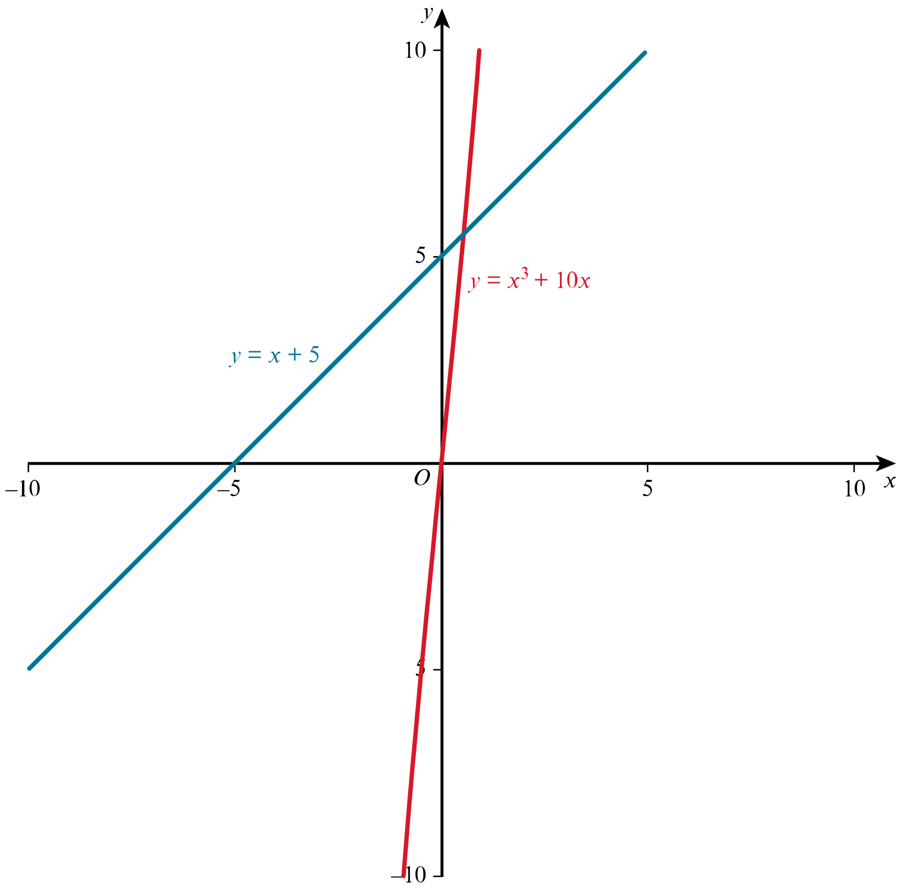
b Using $x_{n+1} = e^{-3x_n} + 0.5$ with $x_1 = 0.5$:

$$
\begin{aligned}
x_2 &= e^{-1.5} + 0.5 = 0.7231 \\
x_3 &= e^{-2.1694} + 0.5 = 0.6142 \\
x_4 &= e^{-1.8427} + 0.5 = 0.6584 \\
x_5 &= e^{-1.9751} + 0.5 = 0.6387 \\
x_6 &= e^{-1.9162} + 0.5 = 0.6472 \\
x_7 &= e^{-1.9415} + 0.5 = 0.6435 \\
x_8 &= e^{-1.9305} + 0.5 = 0.6451 \\
x_9 &= e^{-1.9352} + 0.5 = 0.6444 \\
x_{10} &= e^{-1.9332} + 0.5 = 0.6447
\end{aligned}
$$

So $\alpha = 0.64$ to 2 decimal places.

4 **a** $x^3 + 10x = x + 5$

Drawing the graphs of $y = x^3 + 10x$ and $y = x + 5$:



There is only one point of intersection between 0 and 1, so there is only one root on this range.

b Use the starting point $x_1 = 0.5$ because this is the midpoint of the interval in which there is known to be a root.

Using $x_{n+1} = \dfrac{5 - x_n{}^3}{9}$ and $x_1 = 0.5$:

$$x_1 = 0.5$$
$$x_2 = \frac{5 - 0.125}{9} = 0.541667$$
$$x_3 = \frac{5 - 0.1589265}{9} = 0.537897$$
$$x_4 = \frac{5 - 0.1556315}{9} = 0.538263$$
$$x_5 = \frac{5 - 0.1559495}{9} = 0.538228$$
$$x_6 = \frac{5 - 0.1559188}{9} = 0.538231$$
$$x_7 = \frac{5 - 0.1559218}{9} = 0.538231$$

So $x = 0.5382$ (to 4 decimal points).

5 **a** Let $f(x) = \cos^{-1}(3x) + x - 1 = 0$ then

$f\left(\frac{\pi}{15}\right) = 0.1008459 > 0$ and

Change of sign indicates that there is a root between $\frac{\pi}{15}$ and $\frac{\pi}{12}$.

b $\cos^{-1} 3x = 1 - x$

$$3x = \cos(1 - x)$$
$$x = \frac{1}{3}\cos(1 - x)$$

c Using $x_{n+1} = \frac{1}{3}\cos(1 - x_n)$ with $x_1 = \frac{\pi}{12}$:

$$x_2 = \frac{1}{3}\cos\left(1 - \frac{\pi}{12}\right) = 0.24656$$
$$x_3 = \frac{1}{3}\cos(1 - 0.24656) = 0.24311$$
$$x_4 = \frac{1}{3}\cos(1 - 0.24311) = 0.24233$$
$$x_5 = \frac{1}{3}\cos(1 - 0.24233) = 0.24215$$
$$x_6 = \frac{1}{3}\cos(1 - 0.24215) = 0.24210$$

So $\alpha = 0.242$ to 3 decimal places.

Remember that you can often use a range of starting values. The answers in the coursebook show a number of possibilities. Try some other starting values for yourself and look carefully at which work and which don't work.

6 **a** Using $x_{n+1} = \sqrt{\frac{5 - 2x_n - x_n{}^3}{5}}$ with $x_1 = 1$:

$$x_2 = \sqrt{\frac{5 - 2 - 1}{5}} = 0.6325$$
$$x_3 = \sqrt{\frac{3.4821}{5}} = 0.8345$$
$$x_4 = \sqrt{\frac{2.7498}{5}} = 0.7416$$
$$x_5 = \sqrt{\frac{3.1090}{5}} = 0.7885$$
$$x_6 = \sqrt{\frac{2.9326}{5}} = 0.7658$$
$$x_7 = \sqrt{\frac{3.0191}{5}} = 0.7771$$
$$x_8 = \sqrt{\frac{2.9767}{5}} = 0.7716$$
$$x_9 = \sqrt{\frac{2.9975}{5}} = 0.7743$$
$$x_{10} = \sqrt{\frac{2.9873}{5}} = 0.7730$$

So $\alpha = 0.77$ to 2 decimal places.

b When the sequence converges, $x_n = x_{n+1} = x$.

$$x = \sqrt{\frac{5 - 2x - x^3}{5}}$$
$$x^2 = \frac{5 - 2x - x^3}{5}$$
$$5x^2 = 5 - 2x - x^3$$
$$x^3 + 5x^2 + 2x - 5 = 0$$

7 **a** $x^4 - 1 - x = 0$

$$x^4 = x + 1$$
$$x = \sqrt[4]{x + 1}$$

b Using the result from part **a**:

$$x_{n+1} = \sqrt[4]{1 + x_n}$$

c Using $x_{n+1} = \sqrt[4]{1 + x}$ with $x_1 = 1.5$:

$$x_2 = \sqrt[4]{2.5} = 1.2574$$
$$x_3 = \sqrt[4]{2.2574} = 1.2258$$
$$x_4 = \sqrt[4]{2.2258} = 1.2214$$
$$x_5 = \sqrt[4]{2.2214} = 1.2208$$
$$x_6 = \sqrt[4]{2.2208} = 1.2208$$

So $\alpha = 1.22$ (to 2 decimal places).

8 **a** $x_{n+1} = \sin^{-1}\left(\frac{1}{{x_n}^2}\right)$ or $x_{n+1} = \sqrt{\frac{1}{\sin x_n}}$

b Using $x_{n+1} = \sin^{-1}\left(\frac{1}{{x_n}^2}\right)$:

$$x_1 = 1.5$$
$$x_2 = 0.460554$$

But $\frac{1}{0.460554^2} = 4.714535 > 1$

You cannot take the inverse sine of this number, so the iteration cannot converge.

c Using $x_{n+1} = \sqrt{\frac{1}{\sin x_n}}$ with $x_1 = 1.5$:

$$x_2 = \sqrt{\frac{1}{\sin 1.5}} = 1.00125$$
$$x_3 = \sqrt{\frac{1}{\sin 1.00125}} = 1.08970$$
$$x_4 = \sqrt{\frac{1}{\sin 1.08970}} = 1.06210$$
$$x_5 = \sqrt{\frac{1}{\sin 1.06210}} = 1.07004$$
$$x_6 = \sqrt{\frac{1}{\sin 1.07004}} = 1.06769$$
$$x_7 = \sqrt{\frac{1}{\sin 1.06769}} = 1.06838$$
$$x_8 = \sqrt{\frac{1}{\sin 1.06838}} = 1.06818$$

So $\alpha = 1.068$ to 2 decimal places.

9 **a** Using $x_{n+1} = \ln\left({x_n}^2 + 4\right)$ with $x_1 = 2$:

$$x_2 = \ln(4 + 4) = 2.0794$$
$$x_3 = \ln(4.3241 + 4) = 2.1192$$
$$x_4 = \ln(4.4908 + 4) = 2.1390$$
$$x_5 = \ln(4.5753 + 4) = 2.1489$$
$$x_6 = \ln(4.6177 + 4) = 2.1538$$
$$x_7 = \ln(4.6389 + 4) = 2.1563$$
$$x_8 = \ln(4.6495 + 4) = 2.1575$$
$$x_9 = \ln(4.6548 + 4) = 2.1581$$

So $\alpha = 2.16$ to 2 decimal places.

b

$$x = \ln\left(x^2 + 4\right)$$
$$e^x = x^2 + 4$$
$$x^2 = e^x - 4$$

10 **a**

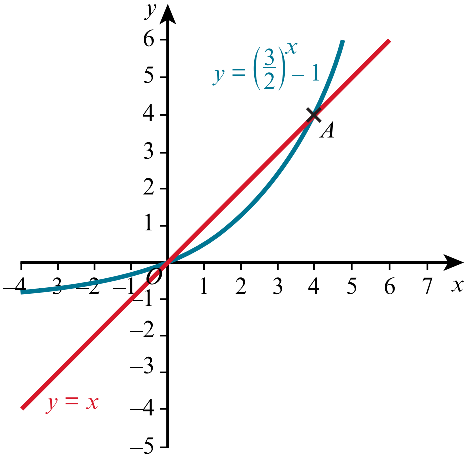
$$\frac{\sin(x-1)}{2x-3} + 1 = 0$$
$$\sin(x-1) + 2x - 3 = 0$$
$$2x = 3 - \sin(x-1)$$
$$x = \frac{3 - \sin(x-1)}{2}$$

b Using $x_{n+1} = \frac{3 - \sin(x_n - 1)}{2}$ with $x_1 = 1.2$:

$$x_2 = \frac{3 - \sin(0.2)}{2} = 1.4007$$
$$x_3 = \frac{3 - \sin(0.4007)}{2} = 1.3050$$
$$x_4 = \frac{3 - \sin(0.3050)}{2} = 1.3499$$
$$x_5 = \frac{3 - \sin(0.3499)}{2} = 1.3286$$
$$x_6 = \frac{3 - \sin(0.3286)}{2} = 1.3386$$
$$x_7 = \frac{3 - \sin(0.3386)}{2} = 1.3339$$
$$x_8 = \frac{3 - \sin(0.3339)}{2} = 1.3361$$
$$x_9 = \frac{3 - \sin(0.3361)}{2} = 1.3351$$
$$x_{10} = \frac{3 - \sin(0.3351)}{2} = 1.3356$$

So $\alpha = 1.34$ to 3 significant figures.

11 a



b $\left(\frac{3}{2}\right)^x - 1 = x$

$$\left(\frac{3}{2}\right)^x = 1 + x$$
$$\ln\left(\frac{3}{2}\right)^x = \ln(1 + x)$$
$$x \ln\left(\frac{3}{2}\right) = \ln(1 + x)$$
$$x = \frac{\ln(1 + x)}{\ln\left(\frac{3}{2}\right)}$$

So a suitable iterative formula is:

$$x_{n+1} = \frac{\ln(1 + x_n)}{\ln\left(\frac{3}{2}\right)}$$

c Using $x_{n+1} = \dfrac{\ln(1 + x_n)}{\ln\left(\frac{3}{2}\right)}$ with $x_1 = 3$:

$x_2 = 3.419023$
$x_3 = 3.664726$
$x_4 = 3.798179$
$x_5 = 3.867747$
$x_6 = 3.903249$
$x_7 = 3.921171$
$x_8 = 3.930169$
$x_9 = 3.934675$
$x_{10} = 3.936928$
$x_{11} = 3.938053$
$x_{12} = 3.938616$

So $x = 3.94$ to 2 decimal places.

The point A is (3.94, 3.94).

$OA^2 = 3.94^2 + 3.94^2$

$OA = \sqrt{3.94^2 + 3.94^2} = 5.6$ to 2 significant figures.

You have to be sure that the decimal place you intend to round to isn't going to change any more. Even if the terms you have written down appear to confirm an answer, it is always worth calculating a few more iterations to check.

12 **a** The height of the cone is the difference between the height of the whole shape and height of the cylinder.

Height of cone $= 33 - 3r$

b

$$5500 = \pi r^2(3r) + \frac{1}{3}\pi r^2(33 - 3r)$$
$$5500 = 3\pi r^3 + \pi r^2(11 - r)$$
$$3\pi r^3 + 11\pi r^2 - \pi r^3 = 5500$$
$$2\pi r^3 + 11\pi r^2 - 5500 = 0$$

c Using the result from part **b**:

$$2\pi r^3 = -11\pi r^2 + 5500$$
$$r^3 = \frac{5500 - 11\pi r^2}{2\pi}$$
$$r = \sqrt[3]{\frac{5500 - 11\pi r^2}{2\pi}}$$

d Using $r_{n+1} = \sqrt[3]{\frac{5500 - 11\pi {r_n}^2}{2\pi}}$ with $x_1 = 8$:

$x_2 = 8.05869$
$x_3 = 8.03200$
$x_4 = 8.04419$
$x_5 = 8.03863$
$x_6 = 8.04117$
$x_7 = 8.04001$
$x_8 = 8.04054$
$x_9 = 8.04030$
$x_{10} = 8.04041$
$x_{11} = 8.04036$
$x_{12} = 8.04038$

So $\alpha = 8.040$ to 3 decimal places.

e The value in part **d** gives the radius of the cone that you would need to choose in order to get the specified volume of $5500\,\text{cm}^3$.

13 $x^3 - 7x^2 + 1 = 0$

$$7x^2 = x^3 + 1$$
$$x^2 = \sqrt{\frac{x^3+1}{7}}$$
$$x_{n+1}^2 = \sqrt{\frac{{x_n}^3+1}{7}}$$
$$x_1 = 0.5$$
$$x_2 = 0.4009$$
$$x_3 = 0.3900$$
$$x_4 = 0.3890$$
$$x_5 = 0.3889$$
$$x_6 = 0.3889$$

$\alpha = 0.39$ (to 2 decimal places)

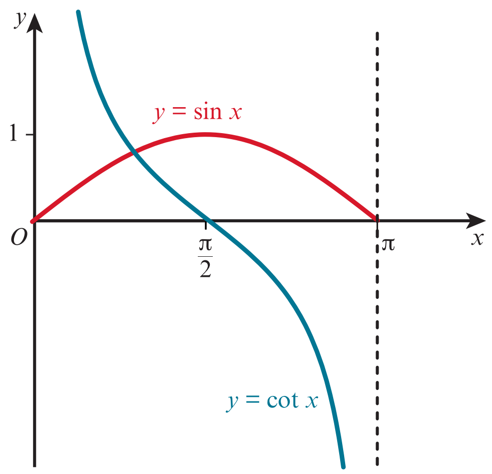
or

$x^3 - 7x^2 + 1 = 0$

$$x^3 = 7x^2 - 1$$
$$x = \sqrt[3]{7x^2 - 1}$$
$$x_{n+1} = \sqrt[3]{7{x_n}^2 - 1}$$
$$x_1 = 6.5$$
$$x_2 = 6.6551$$
$$x_3 = 6.7608$$
$$x_4 = 6.8325$$
$$x_5 = 6.8808$$
$$x_6 = 6.9134$$
$$x_7 = 6.9352$$
$$x_8 = 6.9498$$
$$x_9 = 6.9596$$
$$x_{10} = 6.9662$$
$$x_{11} = 6.9706$$
$$x_{12} = 6.9735$$
$$x_{13} = 6.9755$$
$$x_{14} = 6.9768$$
$$x_{15} = 6.9777$$

$\beta = 6.98$ (to 2 decimal places)

14 **a**



There is an intersection between $x = 0$ and $x = \frac{\pi}{2}$, so there is a root on this domain.

b

$$\cot x = \sin x$$
$$\frac{\cos x}{\sin x} = \sin x$$
$$\cos x = \sin^2 x$$
$$\sin x = \sqrt{\cos x}$$
$$x = \sin^{-1}\sqrt{\cos x}$$

c Using $x_{n+1} = \sin^{-1}\sqrt{\cos x_n}$ with $x_1 = 0.9$:

$$
\begin{aligned}
x_2 &= 0.9082\\
x_3 &= 0.9016\\
x_4 &= 0.9070\\
x_5 &= 0.9026\\
x_6 &= 0.9061\\
x_7 &= 0.9033\\
x_8 &= 0.9056\\
x_9 &= 0.9037\\
x_{10} &= 0.9052\\
x_{11} &= 0.9040\\
x_{12} &= 0.9050\\
x_{13} &= 0.9042
\end{aligned}
$$

So $\alpha = 0.90$ to 2 decimal places.

d $\dfrac{1}{\tan x} = \sin x$

$$\tan x = \frac{1}{\sin x}$$

$$x = \tan^{-1}\left(\frac{1}{\sin x}\right)$$

Using $x_{n+1} = \tan^{-1}\left(\dfrac{1}{\sin x_n}\right)$ with $x_1 = 0.9$:

$$
\begin{aligned}
x_2 &= 0.9063\\
x_3 &= 0.9039\\
x_4 &= 0.9048\\
x_5 &= 0.9045\\
x_6 &= 0.9046\\
x_7 &= 0.9045
\end{aligned}
$$

So $\alpha = 0.90$ to 2 decimal places.

> You can often find several possible iterative formulae and some are much better than others. Convergence can take a long time, as it did in Question **13**, or it can be very fast, as in Question **14 d**. Some iterative processes, such as the Newton–Raphson method, are incredibly quick. Try looking this method up.

EXERCISE 6C

1 **a**

$$y = \frac{e^{2x} + x}{x^3}$$

$$\frac{dy}{dx} = \frac{x^3(2e^{2x} + 1) - (e^{2x} + x)(3x^2)}{x^6}$$

At the stationary point, $\frac{dy}{dx} = 0$:

$$\begin{aligned} x^3(2e^{2x} + 1) - (e^{2x} + x)(3x^2) &= 0 \\ 2x^3e^{2x} + x^3 - 3x^2e^{2x} - 3x^3 &= 0 \\ 2x^3e^{2x} - 2x^3 - 3x^2e^{2x} &= 0 \\ \left(2x^3 - 3x^2\right)e^{2x} &= 2x^3 \\ 2x^3 - 3x^2 &= \frac{2x^3}{e^{2x}} \\ 2x^3 &= 3x^2 + \frac{2x^3}{e^{2x}} \\ x &= \frac{3x^2}{2x^2} + \frac{x}{e^{2x}} \\ x &= \frac{3}{2} + \frac{x}{e^{2x}} \end{aligned}$$

b

> It is worth noting that you are not given a starting point for this iteration. Try different values of x_1 and see how quickly each brings you to the answer. The answers given in the coursebook (and here) will be different if you try a different starting point.

Using $x_{n+1} = \frac{3}{2} + \frac{x_n}{e^{2x_n}}$ with $x_1 = 1.5$:

$$\begin{aligned} x_2 &= 1.574681 \\ x_3 &= 1.567522 \\ x_4 &= 1.568184 \\ x_5 &= 1.568122 \\ x_6 &= 1.568128 \end{aligned}$$

So the x-coordinate of this stationary point is 1.5681 to 4 decimal places.

2 **a**

$$x = t^2 + 6$$

$$\frac{dx}{dt} = 2t$$

$$y = t^4 - t^3 - 5t$$

$$\frac{dy}{dt} = 4t^3 - 3t^2 - 5$$

$$\frac{dy}{dx} = \frac{4t^3 - 3t^2 - 5}{2t}$$

At the stationary point, $\frac{dy}{dx} = 0$:

$$\begin{aligned} 4t^3 - 3t^2 - 5 &= 0 \\ 4t^3 &= 3t^2 + 5 \\ t^3 &= \frac{3t^2 + 5}{4} \\ t &= \sqrt[3]{\frac{3t^2 + 5}{4}} \end{aligned}$$

b Using $t_{n+1} = \sqrt[3]{\frac{3t_n{}^2 + 5}{4}}$ with $x_1 = 1.5$:

$$
\begin{aligned}
x_2 &= 1.432164\\
x_3 &= 1.407497\\
x_4 &= 1.398602\\
x_5 &= 1.395404\\
x_6 &= 1.394256\\
x_7 &= 1.393844\\
x_8 &= 1.393696\\
x_9 &= 1.393643\\
x_{10} &= 1.393624
\end{aligned}
$$

So $t = 1.394$ to 3 decimal places.

c At the stationary point, $t = 1.394$:

$x = t^2 + 6 = 7.94$

$x = 8$ to 1 significant figure.

$y = t^4 - t^3 - 5t = -5.903$

$y = -6$ to 1 significant figure.

The point is $(8, \ -6)$ to 1 significant figure.

3 **a** Shaded segment area $= \dfrac{1}{2}r^2(2\theta) - \dfrac{1}{2}r^2 \sin 2\theta = r^2\theta - \dfrac{1}{2}r^2 \sin 2\theta$

$AB = 2r\cos\theta$

Area of triangle $= \dfrac{1}{2}(2r\cos\theta \times 2r\sin\theta) = 2r^2 \sin\theta\cos\theta$

Using the fact that the area of the triangle is 2 times the area of the shaded segment:

$$
\begin{aligned}
2r^2 \sin\theta\cos\theta &= 2\left(r^2\theta - \frac{1}{2}r^2 \sin 2\theta\right)\\
r^2 \sin 2\theta &= 2r^2\theta - r^2 \sin 2\theta\\
2r^2 \sin 2\theta &= 2r^2\theta\\
\sin 2\theta &= \theta
\end{aligned}
$$

b Using $\theta_{n+1} = \sin 2\theta_n$ with $\theta_1 = \dfrac{\pi}{4}$:

$$
\begin{aligned}
\theta_2 &= 1\\
\theta_3 &= 0.9093\\
\theta_4 &= 0.9695\\
\theta_5 &= 0.9330\\
\theta_6 &= 0.9567\\
\theta_7 &= 0.9419\\
\theta_8 &= 0.9514\\
\theta_9 &= 0.9454\\
\theta_{10} &= 0.9493\\
\theta_{11} &= 0.9468
\end{aligned}
$$

So the root has a value of 0.95 to 2 decimal places.

4 **a**

$$
\begin{aligned}
y &= x^2 \cos 4x\\
\frac{dy}{dx} &= 2x\cos 4x - 4x^2 \sin 4x
\end{aligned}
$$

At point P, $\dfrac{dy}{dx} = 0$:

$$
\begin{aligned}
2x\cos 4x - 4x^2\sin 4x &= 0\\
4x^2 \sin 4x &= 2x\cos 4x\\
\frac{4x^2 \sin 4x}{\cos 4x} &= 2x\\
4x^2 \tan 4x &= 2x
\end{aligned}
$$

b $4x^2 \tan 4x = 2x$

$$\tan 4x = \frac{2x}{4x^2}$$

$$4x = \tan^{-1}\left(\frac{1}{2x}\right)$$

$$x = \frac{1}{4}\tan^{-1}\left(\frac{1}{2x}\right)$$

c Using $x_{n+1} = \frac{1}{4}\tan^{-1}\left(\frac{1}{2x_n}\right)$ with $x_1 = 0.3$:

$x_2 = 0.2576$
$x_3 = 0.2738$
$x_4 = 0.2675$
$x_5 = 0.2699$
$x_6 = 0.2690$
$x_7 = 0.2693$

So the x-coordinate of P is 0.27 to 2 decimal places.

P3

d $\int_0^{\frac{\pi}{8}} x^2 \cos 4x \, dx$

$$= \left[x^2\left(\frac{1}{4}\sin 4x\right)\right]_0^{\frac{\pi}{8}} - \int_0^{\frac{\pi}{8}} \left(\frac{1}{4}(2x)\sin 4x\right) dx$$

$$= \frac{1}{4}\left(\frac{\pi}{8}\right)^2 \sin\frac{\pi}{2} - 0 - \frac{1}{2}\left\{\int_0^{\frac{\pi}{8}} x \sin 4x \, dx\right\}$$

$$= \frac{\pi^2}{256} - \frac{1}{2}\left\{\left[-\frac{1}{4}x\cos 4x\right]_0^{\frac{\pi}{8}} - \int_0^{\frac{\pi}{8}}\left(-\frac{1}{4}\cos 4x\right) dx\right\}$$

$$= \frac{\pi^2}{256} - \frac{1}{2}(0-0) - \frac{1}{8}\int_0^{\frac{\pi}{8}} \cos 4x \, dx$$

$$= \frac{\pi^2}{256} - \frac{1}{8}\left[\frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{8}}$$

$$= \frac{\pi^2}{256} - \frac{1}{32}$$

END-OF-CHAPTER REVIEW EXERCISE 6

1 a Using $x_{n+1} = \frac{6}{7}\left(x_n + \frac{1}{{x_n}^3}\right)$ with $x_1 = 1.5$:

$x_2 = 1.5397$
$x_3 = 1.5546$
$x_4 = 1.5606$
$x_5 = 1.5632$
$x_6 = 1.5643$
$x_7 = 1.5647$
$x_8 = 1.5649$
$x_9 = 1.5650$
$x_{10} = 1.5651$

So $\alpha = 1.57$ to 2 decimal places.

b $x = \frac{6}{7}\left(x + \frac{1}{x^3}\right)$

$7x = 6x + \frac{6}{x^3}$

$x = \frac{6}{x^3}$

$x^4 = 6$

$\alpha = \sqrt[4]{6}$

Remember that the sequence converges to a point where $x_{n+1} = x_n = x$.

2 a $f(x) = x^3 + \frac{x}{2} - 8$

$f(1) = 1 + \frac{1}{2} - 8 < 0$

$f(2) = 8 + 1 - 8 > 0$

Change of sign indicates that there is a solution between 1 and 2.

b $x^3 + \frac{x}{2} - 8 = 0$

$x^3 = 8 - \frac{x}{2}$

$x^2 = \frac{8}{x} - \frac{1}{2}$

$x = \sqrt{\left(\frac{8}{x} - \frac{1}{2}\right)}$

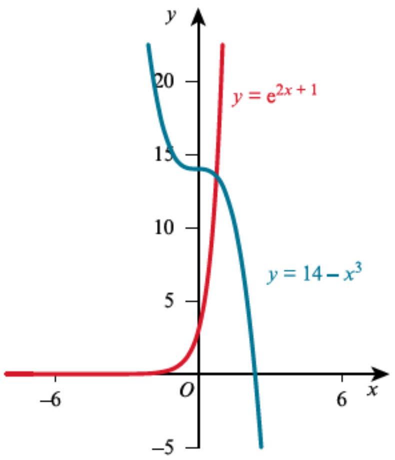
The two equations are equivalent, so if the sequence converges, it converges to the root of the original equation.

c Using $x_{n+1} = \sqrt{\frac{8}{x_n} - \frac{1}{2}}$ with $x_1 = 1.5$:

$x_2 = 2.1985$
$x_3 = 1.7717$
$x_4 = 2.0039$
$x_5 = 1.8688$
$x_6 = 1.9445$
$x_7 = 1.9011$
$x_8 = 1.9256$
$x_9 = 1.9117$
$x_{10} = 1.9196$

So $\alpha = 1.9$ to 1 decimal place.

3 a Draw the graphs of $y = e^{2x+1}$ and $y = 14 - x^3$.



The graphs only intersect at one point, so there is only one real root.

b $f(x) = e^{2x+1} - 14 + x^3$

$f(0.5) = e^2 - 14 + 0.125 < 0$

$f(1) = e^3 - 14 + 1 > 0$

Change of sign indicates that there is a solution between 0.5 and 1.

c

> Before you start to work on the algebra, consider the form of your target equation carefully. You can see that logarithms are required, which makes the path clearer!

$$e^{2x+1} = 14 - x^3$$
$$2x + 1 = \ln(14 - x^3)$$
$$2x = \ln(14 - x^3) - 1$$
$$x = \frac{\ln(14 - x^3) - 1}{2}$$

d Using $x_{n+1} = \dfrac{\ln\left(14 - x^3\right) - 1}{2}$ with $x_1 = 0.75$:

$x_2 = 0.804230$

$x_3 = 0.800597$

$x_4 = 0.800858$

$x_5 = 0.800839$

$x_6 = 0.800840$

$x_7 = 0.800840$

So the root is 0.8008 to 4 decimal places.

4 **a** Using $x_{n+1} = \dfrac{x_n\left(1 + \sec^2 x_n\right) - \tan x_n}{\sec^2 x_n - 1}$ with $x_1 = 1$:

$x_2 = 1.1825$

$x_3 = 1.1692$

$x_4 = 1.1662$

$x_5 = 1.1657$

$x_6 = 1.1656$

So $\alpha = 1.17$ to 2 decimal places.

b

$$x = \frac{x(1 + \sec^2 x) - \tan x}{\sec^2 x - 1}$$
$$x(\sec^2 x - 1) = x(1 + \sec^2 x) - \tan x$$
$$x\sec^2 x - x = x + x\sec^2 x - \tan x$$
$$\tan x - 2x = x\sec^2 x - x\sec^2 x = 0$$

5 **a** $\dfrac{\ln x^2 - 6}{2} = x - 5$

$2\ln x - 6 = 2x - 10$

$2x\ln x + 4 = 2x$

$x = \ln x + 2$

The two equations are equivalent, so if the sequence converges, it converges to the root of the original equation.

b $\mathrm{f}(x) = \ln x + 2 - x$

$\mathrm{f}(2) = \ln 2 + 0 > 0$

$\mathrm{f}(3) = \ln 3 + 2 - 3 = 0.09861... > 0$

$\mathrm{f}(4) = \ln 4 + 2 - 4 = -0.6137... < 0$

Change of sign indicates a root between 3 and 4.

So $p = 3$ and $q = 4$.

c Using $x_{n+1} = \dfrac{\ln {x_n}^2 - 6}{2} + 5$ with $x_1 = 3$:

$x_2 = 3.09861$

$x_3 = 3.13095$

$x_4 = 3.14134$

$x_5 = 3.14465$

$x_6 = 3.14570$

$x_7 = 3.14604$

$x_8 = 3.14614$

$x_9 = 3.14618$

So $\alpha = 3.146$ to 3 decimal places.

d $5^y - 2 = y\ln 5$

$5^y - 2 = \ln 5^y$

$5^y = \ln 5^y + 2$

Letting $x = 5^y$:

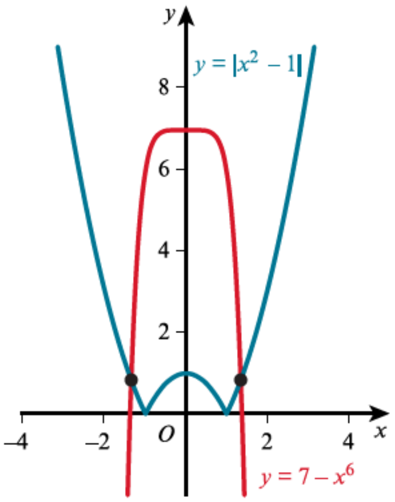
$x = \ln x + 2$

Using the result from part **c**:

$5^y = 3.146$

$y = \log_5 3.146 = 0.71$

6 **a** Draw the graphs of $y = 7 - x^6$ and $y = |x^2 - 1|$.



The equations of both curves only include even powers of x. This means that both curves will be symmetrical about the y-axis.

The graphs intersect at two points, so the equation has two real roots. One of the roots is positive. This root is α, which lies between 1 and 2.

b From the graph in part **a** you can see that the intersection occurs when

$$\begin{aligned} 7 - x^6 &= x^2 - 1 \\ x^6 &= 8 - x^2 \\ x^5 &= \frac{8}{x} - x \\ x &= \sqrt[5]{\frac{8}{x} - x} \end{aligned}$$

c Using $x_{n+1} = \sqrt[5]{\frac{8}{x_n} - x_n}$ with $x_1 = 1.5$:

$x_2 = 1.3083$
$x_3 = 1.3689$
$x_4 = 1.3495$
$x_5 = 1.3557$
$x_6 = 1.3537$
$x_7 = 1.3543$
$x_8 = 1.3541$

So $\alpha = 1.35$ to 3 significant figures.

d The graph has reflection symmetry in the y-axis, so $\beta = -\alpha = -1.35$.

7 **a**

$$\begin{aligned} \int_0^a \left(\frac{1}{3x+5} + 2e^{6x}\right) dx &= 0.6 \\ \left[\frac{1}{3}\ln(3x+5) + \frac{1}{3}e^{6x}\right]_0^a &= 0.6 \\ \frac{1}{3}\ln(3a+5) + \frac{1}{3}e^{6a} - \frac{1}{3}\ln 5 - \frac{1}{3} &= 0.6 \end{aligned}$$

$$e^{6a} = 1 + 1.8 + \ln 5 - \ln(3a+5)$$

$$6a = \ln\left[2.8 + \ln\left(\frac{5}{3a+5}\right)\right]$$

$$a = \frac{1}{6}\left[2.8 + \ln\left(\frac{5}{3a+5}\right)\right]$$

b Using $a_{n+1} = \frac{1}{6}\ln\left(2.8 + \ln\left(\frac{5}{3a_n+5}\right)\right)$ with $a_1 = 0.2$:

$a_2 = 0.164717$
$a_3 = 0.165897$
$a_4 = 0.165857$
$a_5 = 0.165858$

So $\alpha = 0.166$ to 3 decimal places.

8 **a**

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{1}{2}(t+1)^{-\frac{1}{2}}}{-t^{-2}}$$

$$= -\frac{t^2}{2\sqrt{t+1}}$$

At P:

$$\frac{dy}{dx} = -\frac{p^2}{2\sqrt{p+1}} = -0.4$$
$$p^2 = 0.8\sqrt{p+1}$$
$$p^2 = \left(\frac{4}{5}\right)\sqrt{p+1}$$
$$p^4 = \left(\frac{4}{5}\right)^2(p+1)$$
$$p = \sqrt{\left(\frac{4}{5}\right)}(p+1)^{\frac{1}{4}}$$
$$p = \frac{2}{\sqrt{5}}(p+1)^{\frac{1}{4}}$$
$$p = \frac{2\sqrt{5}}{5}(p+1)^{\frac{1}{4}}$$

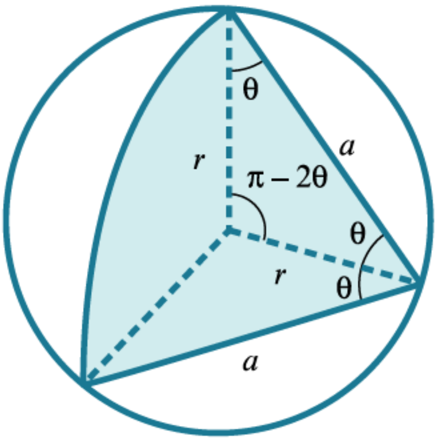
Remember to rationalise the denominator.

b Using $p_{n+1} = \frac{2\sqrt{5}}{5}(p_n+1)^{\frac{1}{4}}$ with $p_1 = 1$:

$p_2 = 1.06366$
$p_3 = 1.07202$
$p_4 = 1.07311$
$p_5 = 1.07325$
$p_6 = 1.07327$

So $p = 1.073$ to 3 decimal places.

9 a



Using the sine rule:

Remember that $\sin(\pi - \theta) = \sin\theta$. This is because the graph of $y = \sin\theta$ has a line of symmetry at $\theta = \frac{\pi}{2}$.

$$\frac{r}{\sin\theta} = \frac{a}{\sin(\pi - 2\theta)}$$
$$\frac{r}{\sin\theta} = \frac{a}{\sin 2\theta}$$
$$r\sin 2\theta = a\sin\theta$$
$$2r\sin\theta\cos\theta = a\sin\theta$$
$$a = 2r\cos\theta$$

Using the fact that the area of the shaded region is equal to $\frac{3}{8}$ of the area of the circle:

$$\frac{\frac{1}{2}a^2(2\theta)}{\pi r^2} = \frac{3}{8}$$

$$\frac{8r^2\theta\cos^2\theta}{2\pi r^2} = \frac{3}{8}$$

$$\cos^2\theta = \frac{3\pi}{32\theta}$$

$$\cos\theta = \sqrt{\frac{3\pi}{32\theta}}$$

$$\theta = \cos^{-1}\sqrt{\frac{3\pi}{32\theta}}$$

b $\mathrm{f}(\theta) = \theta - \cos^{-1}\sqrt{\frac{3\pi}{32\theta}}$

$\mathrm{f}(0.8) = 0.8 - \cos^{-1}\sqrt{\frac{3\pi}{25.6}} < 0$

$\mathrm{f}(1.2) = 1.2 - \cos^{-1}\sqrt{\frac{3\pi}{38.4}} > 0$

Change of sign indicates that θ lies between 0.8 and 1.2.

c Using $\theta_{n+1} = \cos^{-1}\sqrt{\frac{3\pi}{32\theta_n}}$ with $\theta_1 = 1$:

$\theta_2 = 0.99715$
$\theta_3 = 0.99622$
$\theta_4 = 0.99592$
$\theta_5 = 0.99582$
$\theta_6 = 0.99579$
$\theta_7 = 0.99578$
$\theta_8 = 0.99578$

So $\theta = 0.996$ to 3 decimal places.

10 **a** $\mathrm{f}'(x) = \mathrm{e}^{x-2}$

$\mathrm{g}'(x) = \sin x$

Letting $\mathrm{h}(x) = \mathrm{f}'(x) - \mathrm{g}'(x)$:

$\mathrm{h}(x) = \mathrm{e}^{x-2} - \sin x$

When $x = p$, $\mathrm{h}(p) = 0$.

$\mathrm{h}(0.155) = 0.00364 > 0$

$\mathrm{h}(0.165) = -0.00463 < 0$

Change of sign indicates the presence of a root between 0.155 and 0.165, which means that the root is 0.16 to 2 decimal places.

b $\mathrm{f}'(q) = \mathrm{g}'(q)$

$\mathrm{e}^{q-2} = \sin q$

$q - 2 = \ln(\sin q)$

$q = 2 + \ln(\sin q)$

c Using $q_{n+1} = 2 + \ln(\sin q_n)$ with $q_1 = 2$:

Using $q_{n+1} = 2 + \ln(\sin q_n)$ with $q_1 = 2$:

$q_2 = 1.9049$
$q_3 = 1.9431$
$q_4 = 1.9290$
$q_5 = 1.9344$
$q_6 = 1.9324$
$q_7 = 1.9332$
$q_8 = 1.9329$
$q_9 = 1.9330$
$q_{10} = 1.9329$
$q_{11} = 1.9329$

So $q = 1.93$ to 2 decimal places.

11 **a** $x = \cos x + \sin x$

$= R\cos(x - a)$

$= R\cos x \cos a + R\sin x \sin a$

Equating coefficients of $\cos x$: $R\cos a = 1$ .... [1]

Equating coefficients of $\sin x$: $R\sin a = 1$ .... [2]

Dividing [2] by [1]:

$\tan a = 1$

$a = \frac{\pi}{4}$

$R^2 = 1^2 + 1^2 = 2$

$R = \sqrt{2}$

$x = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$

b Using $x_{n+1} = \sqrt{2}\cos\left(x_n - \frac{\pi}{4}\right)$ with $x_1 = 1.2$:

$x_2 = 1.29440$

$x_3 = 1.23494$

$x_4 = 1.27371$

$x_5 = 1.24893$

$x_6 = 1.26498$

$x_7 = 1.25467$

$x_8 = 1.26133$

$x_9 = 1.25704$

$x_{10} = 1.25981$

$x_{11} = 1.25803$

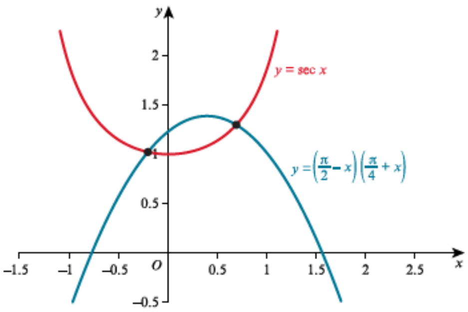
$x_{12} = 1.25918$

$x_{13} = 1.25844$

$x_{14} = 1.25892$

So $\alpha = 1.26$ to 2 decimal places.

P3 **12** **a**



There are two points of intersection in the given interval, so there are two real roots.

b

$$\sec x = \left(\frac{\pi}{2} - x\right)\left(\frac{\pi}{4} + x\right)$$

$$-x^2 + \frac{\pi}{4}x + \frac{\pi^2}{8} = \sec x$$

$$x^2 = \frac{\pi}{4}x + \frac{\pi^2}{8} - \sec x$$

$$= \frac{2\pi x + \pi^2 - 8\sec x}{8}$$

$$x = \sqrt{\frac{2\pi x + \pi^2 - 8\sec x}{8}}$$

c

$$\mathrm{f}(x) = \sec x - \left(\frac{\pi}{2} - x\right)\left(\frac{\pi}{4} + x\right)$$

$$\mathrm{f}(-0.215) = \sec(-0.215) - \left(\frac{\pi}{2} + 0.215\right)\left(\frac{\pi}{4} - 0.215\right) > 0$$

$$\mathrm{f}(-0.205) = \sec(-0.205) - \left(\frac{\pi}{2} + 0.205\right)\left(\frac{\pi}{4} - 0.205\right) < 0$$

Change of sign indicates the presence of a root between -0.205 and -0.215, which means that the root is -0.21 to 2 decimal places.

d Using $x_{n+1} = \sqrt{\dfrac{2\pi x + \pi^2 - 8\sec x_n}{8}}$ with $x_1 = 1$:

$x_2 = 0.4102$
$x_3 = 0.6822$
$x_4 = 0.6936$
$x_5 = 0.6913$
$x_6 = 0.6918$
$x_7 = 0.6917$
$x_8 = 0.6918$

So $\beta = 0.69$ to 2 decimal places.

13 **a**

$$u = \sin x$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$$
$$\mathrm{d}u = \cos x\,\mathrm{d}x$$
$$\cos^2 x = 1 - \sin^2 x$$
$$\cos^2 x = 1 - u^2$$
$$\cos^3 x\,\mathrm{d}x = \left(1 - u^2\right)\mathrm{d}u$$

When $x = 0$, $u = 0$

When $x = \alpha$, $u = \sin\alpha$

$$\int_0^\alpha \cos^3 x\,\mathrm{d}x = 0.3$$
$$\int_0^{\sin\alpha} \left(1 - u^2\right)\mathrm{d}u = 0.3$$
$$\left[u - \frac{1}{3}u^3\right]_0^{\sin\alpha} = 0.3$$
$$\sin\alpha - \frac{1}{3}\sin^3\alpha - 0 + 0 = 0.3$$
$$3\sin\alpha - \sin^3\alpha = 0.9$$
$$\sin\alpha\left(3 - \sin^2\alpha\right) = 0.9$$
$$\sin\alpha = \frac{0.9}{3 - \sin^2\alpha}$$

b Using $\alpha_{n+1} = \sin^{-1}\left(\dfrac{0.9}{3 - \sin^2 x_n}\right)$ with $\alpha_1 = 0.2$:

$\alpha_2 = 0.30889$
$\alpha_3 = 0.31470$
$\alpha_4 = 0.31509$
$\alpha_5 = 0.31511$

So $a = 0.315$ to 3 significant figures.

14 **a**

> Remember that integration by parts requires a product of two terms in the integrand.

$$\int_1^a \ln x\,\mathrm{d}x$$
$$= \int_1^a 1(\ln x)\,\mathrm{d}x$$
$$= [x\ln x]_1^a - \int_1^a x\left(\frac{1}{x}\right)\mathrm{d}x$$
$$= a\ln a - 0 - \int_1^a 1\,\mathrm{d}x$$
$$= a\ln a - [x]_1^a$$
$$= a\ln a - (a - 1)$$
$$= a\ln a - a + 1$$

Using the fact that $\int_1^a \ln x \, dx = 5$:

$$\begin{aligned} a \ln a - a + 1 &= 5 \\ a \ln a &= 4 + a \\ a &= \frac{4 + a}{\ln a} \end{aligned}$$

b Using $a_{n+1} = \dfrac{4 + a_n}{\ln a_n}$ with $a_1 = 5$:

$a_2 = 5.59201$
$a_3 = 5.57241$
$a_4 = 5.57239$
$a_5 = 5.57239$
So $a = 5.572$ to 3 decimal places.

CROSS-TOPIC REVIEW EXERCISE 2

1 $y = \frac{1+x}{1+2x}$

Using the quotient rule:

$$\frac{dy}{dx} = \frac{(1+2x)(1)-(1+x)(2)}{(1+2x)^2}$$
$$= \frac{1+2x-2-2x}{(1+2x)^2}$$
$$= -\frac{1}{(1+2x)^2}$$

$x > -\frac{1}{2}$

so $(1+2x)^2 > 0$

and $\frac{dy}{dx} = -\frac{1}{(1+2x)^2} < 0$ for $x > -\frac{1}{2}$

> It is fine to 'explain' using inequalities. Words are not always necessary in mathematics.

2 $y = \frac{1}{2}\tan 2x$

Using the chain rule:

$$\frac{dx}{dx} = \frac{1}{2}\left(\sec^2 2x\right)(2)$$
$$= \sec^2 2x$$

When the gradient is 4, $\frac{dy}{dx} = 4$:

$\sec^2 2x = 4$

$\cos^2 2x = \frac{1}{4}$

$\cos 2x = \frac{1}{2}$ or $-\frac{1}{2}$

$2x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

$x = \frac{\pi}{6}, \frac{\pi}{3}$

3 **i** $y = 2t^{-1}$

$$\frac{dy}{dt} = -2t^{-2} = -\frac{2}{t^2}$$

$x = \ln(1-2t)$

$$\frac{dx}{dt} = \frac{-2}{1-2t}$$
$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= -\frac{2}{t^2} \times \frac{1-2t}{-2}$$
$$= \frac{1-2t}{t^2}$$

> Remember that $\frac{dt}{dx} = \frac{1}{\left(\frac{dx}{dt}\right)}$.

ii When the gradient is 3, $\frac{dy}{dx} = 3$:

$$\frac{1-2t}{t^2} = 3$$
$$1-2t = 3t^2$$
$$3t^2 + 2t - 1 = 0$$
$$(3t-1)(t+1) = 0$$
$t = \frac{1}{3}$ or $t = -1$

But $t < 0$

so $t = -1$

$x = \ln 3$

$y = -2$

So the coordinates of the only point on the curve at which the gradient is 3 are $(\ln 3, -2)$.

4 i Differentiating with respect to x, using the chain and product rules:

$$x^2y + y^2 = 6x$$
$$2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 6$$
$$\frac{dy}{dx}(x^2 + 2y) = 6 - 2xy$$
$$\frac{dy}{dx} = \frac{6-2xy}{x^2+2y}$$

ii At the point (1, 2):

$$\frac{dy}{dx} = \frac{6-2xy}{x^2+2y} = \frac{6-2(1)(2)}{1^2+2(2)} = \frac{2}{5}$$

Equation of tangent:

$$y - 2 = \frac{2}{5}(x-1)$$
$$5y - 10 = 2x - 2$$
$$2x - 5y + 8 = 0$$

P2 5 i Set out your different ordinates carefully in a table, as shown here.

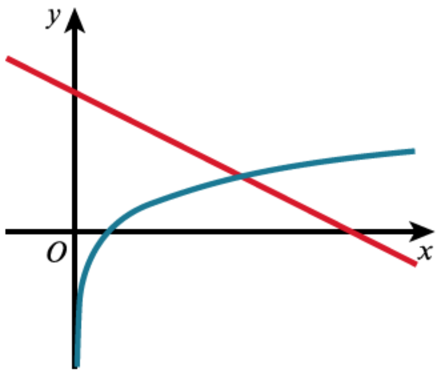
$$h = \frac{1-0}{2} = \frac{1}{2}$$

x	0	$\frac{1}{2}$	1
y	$\frac{1}{6+2e^0}$	$\frac{1}{6+2e^{0.5}}$	$\frac{1}{6+2e^1}$

$$\int_0^1 \frac{1}{6+2e^x}dx \approx \frac{h}{2}[y_0 + y_2 + 2y_1]$$
$$= \frac{1}{4}\left[\frac{1}{6+2e^0} + \frac{1}{6+2e^1} + \frac{2}{6+2e^{0.5}}\right]$$
$$= 0.11 \text{ (to 2 decimal places)}$$

ii $$\int \frac{(e^x-2)^2}{e^{2x}}dx$$
$$= \int \left(\frac{e^{2x} - 4e^x + 4}{e^{2x}}\right) dx$$
$$= \int (1 - 4e^{-x} + 4e^{-2x})\, dx$$
$$= x + 4e^{-x} - 2e^{-2x} + c$$

6 i Sketching the graphs of $y = \ln x$ and $y = 4 - \frac{1}{2}x$:



Note that here is only one point of intersection, so the given equation has only one real root.

ii $\quad f(x) = \ln x - 4 + \frac{1}{2}x$

$f(4.5) = -0.2459 < 0$

$f(5) = 0.1093 > 0$

Change of sign indicates the presence of a root between 4.5 and 5.

So, $4.5 < \alpha < 5.0$.

iii Using $x_{n+1} = 8 - 2\ln x_n$ with $x_0 = 4.75$:

$x_1 = 4.8837$

$x_2 = 4.8282$

$x_3 = 4.8511$

$x_4 = 4.8416$

$x_5 = 4.8455$

$x_6 = 4.8439$

$x_7 = 4.8446$

$x_8 = 4.8443$

$x_9 = 4.8444$

So, $\alpha = 4.84$ (to 2 decimal places)

7 **i**

$x = (\cos t)^{-3}$

$$\frac{dx}{dt} = -3(\cos t)^{-4}(-\sin t)$$

$$= \frac{3\sin t}{\cos^4 t}$$

$y = \tan^3 t$

$$\frac{dy}{dt} = 3\tan^2 t\sec^2 t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 3\tan^2 t\ \sec^2 t \times \left(\frac{1}{3\sin t}\cos^4 t\right)$$

$$= \frac{\tan^2 t\cos^2 t}{\sin t}$$

$$= \frac{\sin^2 t\cos^2 t}{\sin t\cos^2 t}$$

$$= \sin t$$

ii Gradient $= \sin t$ at the point $\left(\frac{1}{\cos^3 t}, \tan^3 t\right)$.

Equation of tangent:

$$y - \tan^3 t = \sin t\left(x - \frac{1}{\cos^3 t}\right)$$

$$\begin{aligned}
y &= \tan^3 t + x\sin t - \frac{\sin t}{\cos^3 t}\\
&= x\sin t + \frac{\sin^3 t}{\cos^3 t} - \frac{\sin t}{\cos^3 t}\\
&= x\sin t + \frac{\sin t(\sin^2 t - 1)}{\cos^3 t}\\
&= x\sin t - \frac{\sin t\cos^2 t}{\cos^3 t}\\
&= x\sin t - \frac{\sin t}{\cos t}\\
y &= x\sin t - \tan t
\end{aligned}$$

8 **i**

$$\begin{aligned}
\int_{-a}^{a}(4e^{2x} + 5)\,dx &= 100\\
\left[2e^{2x} + 5x\right]_{-a}^{a} &= 100\\
2e^{2a} + 5a - \left(2e^{-2a} - 5a\right) &= 100\\
2e^{2a} + 10a - 2e^{-2a} &= 100\\
e^{2a} &= 50 + e^{-2a} - 5a\\
2a &= \ln\left(50 + e^{-2a} - 5a\right)\\
a &= \frac{1}{2}\ln\left(50 + e^{-2a} - 5a\right)
\end{aligned}$$

ii $f(a) = \frac{1}{2}\ln\left(50 + e^{-2a} - 5a\right) - a$

$f(1) = 0.90483 > 0$

$f(2) = -0.15533 < 0$

Change of sign indicates the presence of a root between 1 and 2.

So, $1 < \alpha < 2$.

Using $a_{n+1} = \frac{1}{2}\ln\left(50 + e^{-2a_n} - 5a_n\right)$ with $x_0 = 1.5$:

$x_1 = 1.87534$

$x_2 = 1.85246$

$x_3 = 1.85388$

$x_4 = 1.85379$

$x_5 = 1.85380$

$x_6 = 1.85380$

So, $\alpha = 1.854$ (to 3 decimal places)

9 **i** $y = e^{-2x}\tan x$

Using the product and chain rules:

$$\frac{dy}{dx} = -2e^{-2x}\tan x + e^{-2x}\sec^2 x$$

but $\tan^2 x + 1 \equiv \sec^2 x$

$$\begin{aligned}
\frac{dy}{dx} &= -2e^{-2x}\tan x + e^{-2x}(\tan^2 x + 1)\\
&= e^{-2x}\left(\tan^2 x + 1 - 2\tan x\right)\\
&= e^{-2x}(\tan^2 x - 2\tan x + 1)\\
&= e^{-2x}(1 - \tan x)^2
\end{aligned}$$

Remember that $(A - B)^2 = A^2 - 2AB + B^2$. You will then recognise how the expression has been factorised in part **i**.

ii

$$\frac{dy}{dx} = e^{-2x}(1 - \tan x)^2$$

$$e^{-2x} > 0$$

$$(1 - \tan x)^2 \geqslant 0$$

So, $\frac{\mathrm{d}y}{\mathrm{d}x} \geqslant 0$

So, the gradient is never negative.

iii $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-2x}(1 - \tan x)^2$

Least value is when

$$\begin{aligned}(1 - \tan x)^2 &= 0\\ \tan x &= 1\\ x &= \frac{\pi}{4}\end{aligned}$$

10 **i** $\tan\theta + \cot\theta$

$$\begin{aligned}&\equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\\ &\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\\ &\equiv \frac{1}{\sin\theta\cos\theta}\\ &\equiv \frac{2}{2\sin\theta\cos\theta}\\ &\equiv \frac{2}{\sin 2\theta}\end{aligned}$$

ii **a** Using the result from part **i**:

$$\begin{aligned}\tan\frac{\pi}{8} + \cot\frac{\pi}{8} &= \frac{2}{\sin\left(2 \times \frac{\pi}{8}\right)}\\ &= \frac{2}{\sin\frac{\pi}{4}}\\ &= \frac{2}{\left(\frac{1}{\sqrt{2}}\right)}\\ &= 2\sqrt{2}\end{aligned}$$

b $\int_0^{\frac{\pi}{2}} \frac{6}{\tan\theta + \cot\theta}\,\mathrm{d}\theta$

$$\begin{aligned}&= \int_0^{\frac{\pi}{2}} 6\left(\frac{\sin 2\theta}{2}\right)\mathrm{d}\theta\\ &= \int_0^{\frac{\pi}{2}} 3\sin 2\theta\,\mathrm{d}\theta\\ &= \left[-\frac{3}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}}\\ &= -\frac{3}{2}\cos\pi + \frac{3}{2}\cos 0\\ &= \frac{3}{2} + \frac{3}{2}\\ &= 3\end{aligned}$$

11 $xy(x - 6y) = 9a^3$

$x^2y - 6xy^2 = 9a^3$

Differentiating with respect to x.

$$\begin{aligned}2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} - 6y^2 - 12xy\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \frac{\mathrm{d}y}{\mathrm{d}x}(x^2 - 12xy) &= 6y^2 - 2xy\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6y^2 - 2xy}{x^2 - 12xy}\\ &= \frac{2y(3y - x)}{x(x - 12y)}\end{aligned}$$

Parallel to the x-axis when $\frac{dy}{dx} = 0$:

$2y(3y - x) = 0$

$y = 0$ or $y = \frac{1}{3}x$

But $y \neq 0$ as this is inconsistent with the equation of the curve.

So, $y = \frac{1}{3}x$.

Substituting into the original equation:

$$x\left(\frac{1}{3}x\right)\left[x - 6\left(\frac{1}{3}x\right)\right] = 9a^3$$

$$x^2(x - 2x) = 27a^3$$

$$-x^3 = 27a^3$$

$$x = -3a$$

$$y = \frac{1}{3}x = -a$$

So, there is only one point and it has coordinates $(-3a, -a)$.

12 **i** $f(x) = \frac{6}{x^2} - x - 1$

$f(1.4) = 0.66122 > 0$

$f(1.6) = -0.25625 < 0$

Change of sign indicates the presence of a root between 1.4 and 1.6.

ii Intersect when

$$\frac{6}{x^2} = x + 1$$

$$x^2(x + 1) = 6$$

$$x^2 = \frac{6}{x + 1}$$

The root lies between 1.4 and 1.6, so it is positive.

$$x = \sqrt{\left(\frac{6}{x + 1}\right)}$$

iii Using $x_{n+1} = \sqrt{\left(\frac{6}{x_n + 1}\right)}$ with $x_0 = 1.5$:

$x_1 = 1.5492$

$x_2 = 1.5342$

$x_3 = 1.5387$

$x_4 = 1.5373$

$x_5 = 1.5378$

$x_6 = 1.5376$

So, the x-coordinate of P is 1.54 to 2 decimal places.

13 **i** $x = e^{2t}$

$\frac{dx}{dt} = 2e^{2t}$

$y = 4t\,e^t$

$\frac{dy}{dt} = 4e^t + 4t\,e^t$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4e^t + 4t\,e^t}{2e^{2t}}$$

$$= \frac{2 + 2t}{e^t}$$

$$= \frac{2(t + 1)}{e^t}$$

ii Remember that a normal is always perpendicular to a tangent at the same point. If you take the product of the gradients of the normal and tangent you get -1.

When $t = 0$:

$$\frac{dy}{dx} = \frac{2(t+1)}{e^t} = \frac{2}{1} = 2$$

Gradient of normal $= -\frac{1}{2}$

At $t = 0$:

$$x = e^0 = 1$$
$$y = 4t\,e^t = 0$$

Equation of normal is:

$$y - 0 = -\frac{1}{2}(x-1)$$
$$y = -\frac{1}{2}x + \frac{1}{2}$$
$$2y = -x + 1$$
$$x + 2y = 1$$

14 i $y = 3\cos 2x - 5\sin x$

$$\frac{dy}{dx} = -6\sin 2x - 5\cos x$$

At $x = \frac{\pi}{6}$:

$$\frac{dy}{dx} = -6\sin\frac{\pi}{3} - 5\cos\frac{\pi}{6}$$
$$= -6\left(\frac{\sqrt{3}}{2}\right) - 5\left(\frac{\sqrt{3}}{2}\right)$$
$$= -\frac{11\sqrt{3}}{2}$$

ii $x^3 + 6xy + y^3 = 21$

Differentiating with respect to x:

$$3x^2 + 6y + 6x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(6x + 3y^2) = -3x^2 - 6y$$
$$\frac{dy}{dx} = \frac{-3x^2 - 6y}{(6x + 3y^2)}$$
$$= -\frac{3x^2 + 6y}{6x + 3y^2}$$

At (1, 2):

$$\frac{dy}{dx} = -\frac{3+12}{6+12}$$
$$= -\frac{15}{18}$$
$$= -\frac{5}{6}$$

15 i $y = \sec x$

$$= \frac{1}{\cos x}$$
$$= (\cos x)^{-1}$$
$$\frac{dy}{dx} = -(\cos x)^{-2}(-\sin x)$$
$$= \frac{\sin x}{\cos^2 x}$$
$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$$
$$= \sec x \tan x$$

ii Notice that you want to get $\sec x + \tan x$ on top of this identity, so you must multiply top and bottom by $\sec x + \tan x$.

$$\frac{1}{\sec x - \tan x}$$
$$\equiv \frac{\sec x + \tan x}{(\sec x - \tan x)(\sec x + \tan x)}$$
$$\equiv \frac{\sec x + \tan x}{\sec^2 x - \tan^2 x} \text{ (Difference of two squares)}$$
$$\equiv \frac{\sec x + \tan x}{1 + \tan^2 x - \tan^2}$$
$$\equiv \sec x + \tan x$$

iii Using the result from part **ii**:

$$\frac{1}{(\sec x - \tan x)^2}$$
$$\equiv (\sec x + \tan x)^2$$
$$\equiv \sec^2 x + 2 \sec x \tan x + \tan^2 x$$
$$\equiv \sec^2 x + \sec^2 x - 1 + 2 \sec x \tan x$$
$$\equiv 2 \sec^2 x - 1 + 2 \sec x \tan x$$

iv Using the result from part **iii**:

$$\int_0^{\frac{\pi}{4}} \frac{1}{(\sec x - \tan x)^2} \, dx$$
$$= \int_0^{\frac{\pi}{4}} \left(2 \sec^2 x - 1 + 2 \sec x \tan x\right) dx$$
$$= [2 \tan x - x + 2 \sec x]_0^{\frac{\pi}{4}}$$
$$= 2 \tan \frac{\pi}{4} - \frac{\pi}{4} + 2 \sec \frac{\pi}{4} - 0 + 0 - 2 \sec 0$$
$$= 2 - \frac{\pi}{4} + \frac{2}{\left(\frac{1}{\sqrt{2}}\right)} - 2$$
$$= 2\sqrt{2} - \frac{\pi}{4}$$
$$= \frac{1}{4}(8\sqrt{2} - \pi)$$

16 **i**

$$x = 2 \tan \theta$$
$$\frac{dx}{d\theta} = 2 \sec^2 \theta$$
$$y = 3 \sin 2\theta$$
$$\frac{dy}{d\theta} = 6 \cos 2\theta$$
$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$
$$= \frac{6 \cos 2\theta}{2 \sec^2 \theta}$$
$$= 3 \cos^2 \theta \cos 2\theta$$
$$= 3 \cos^2 \theta \left(2 \cos^2 \theta - 1\right)$$
$$= 6 \cos^4 \theta - 3 \cos^2 \theta$$

ii Stationary point when $6 \cos^4 \theta - 3 \cos^2 \theta = 0$:

$3 \cos^2 \theta \left(2 \cos^2 \theta - 1\right) = 0$

$\cos \theta = 0$ or $\cos 2\theta = 0$

$\theta = \frac{\pi}{2}$ (out of range) or $2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

$x = 2\tan\frac{\pi}{4} = 2$

$y = 3\sin 2\left(\frac{\pi}{4}\right) = 3\sin\frac{\pi}{2} = 3$

The point is $(2,\ 3)$.

iii At $\left(2\sqrt{3},\ \frac{3}{2}\sqrt{3}\right)$:

$$\begin{aligned} x &= 2\sqrt{3} \\ 2\tan\theta &= 2\sqrt{3} \\ \tan\theta &= \sqrt{3} \\ \theta &= \frac{\pi}{3} \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 6\cos^4\theta - 3\cos^2\theta \\ &= 6\left(\frac{1}{2}\right)^4 - 3\left(\frac{1}{2}\right)^2 \\ &= \frac{6}{16} - \frac{3}{4} \\ &= -\frac{6}{16} \\ &= -\frac{3}{8} \end{aligned}$$

17 **i** $y = 4\mathrm{e}^{\frac{1}{2}x} - 6x + 3$

At the stationary point:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 2\mathrm{e}^{\frac{1}{2}x} - 6 = 0 \\ 2\mathrm{e}^{\frac{1}{2}x} &= 6 \\ \mathrm{e}^{\frac{1}{2}x} &= 3 \\ \frac{1}{2}x &= \ln 3 \\ x &= 2\ln 3 \\ &= \ln 3^2 \\ &= \ln 9 \\ a &= 9 \end{aligned}$$

ii $\int_0^2 \left(4\mathrm{e}^{\frac{1}{2}x} - 6x + 3\right)\mathrm{d}x$

$= \left[8\mathrm{e}^{\frac{1}{2}x} - 3x^2 + 3x\right]_0^2$

$= 8\mathrm{e}^1 - 12 + 6 - (8 - 0 + 0)$

$= 8\mathrm{e} - 14$

18 **i** $x^3 - 3x^2y + y^3 = 3$

Differentiating with respect to x:

$$\begin{aligned} 3x^2 - 6xy - 3x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x}(3y^2 - 3x^2) &= 6xy - 3x^2 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6xy - 3x^2}{3y^2 - 3x^2} \\ &= \frac{2xy - x^2}{y^2 - x^2} \\ &= \frac{x^2 - 2xy}{x^2 - y^2} \end{aligned}$$

Remember that $A - B = -(B - A)$.

ii Parallel to the x-axis when $\frac{dy}{dx} = 0$:

$$x^2 - 2xy = 0$$
$$x(x - 2y) = 0$$
$$x = 0 \text{ or } x = 2y$$

When $x = 0$:

$$\begin{aligned} x^3 - 3x^2y + y^3 &= 3 \\ y^3 &= 3 \\ y &= \sqrt[3]{3} = 1.44 \end{aligned}$$

When $x = 2y$:

$$\begin{aligned} x^3 - 3x^2y + y^3 &= 3 \\ (2y)^3 - 3(2y)^2y + y^3 &= 3 \\ 8y^3 - 12y^3 + y^3 &= 3 \\ -3y^3 &= 3 \\ y &= -1 \\ x &= -2 \end{aligned}$$

The points are $(-2, -1)$ and $(0, 1.44)$.

Check the degree of accuracy required by the question.

19 i

$$\begin{aligned} x &= (2t + 1)^{-2} \\ \frac{dx}{dt} &= -4(2t + 1)^{-3} \\ &= -\frac{4}{(2t + 1)^3} \end{aligned}$$

$$\begin{aligned} y &= (t + 2)^{\frac{1}{2}} \\ \frac{dy}{dt} &= \frac{1}{2}(t + 2)^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{t + 2}} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1}{2\sqrt{t + 2}} \times -\frac{(2t + 1)^3}{4} \\ &= -\frac{(2t + 1)^3}{8\sqrt{t + 2}} \end{aligned}$$

At $x = p$, $\frac{dy}{dx} = -1$:

$$-\frac{(2p + 1)^3}{8\sqrt{p + 2}} = -1$$

$$\begin{aligned} (2p + 1)^3 &= 8\sqrt{p + 2} \\ (2p + 1)^6 &= 64(p + 2) \\ 2p + 1 &= 2(p + 2)^{\frac{1}{6}} \\ 2p &= 2(p + 2)^{\frac{1}{6}} - 1 \\ p &= (p + 2)^{\frac{1}{6}} - \frac{1}{2} \end{aligned}$$

ii Using $p_{n+1} = (p_n + 2)^{\frac{1}{6}} - \frac{1}{2}$ with $p_0 = 0.7$:

$p_1 = 0.68003$
$p_2 = 0.67857$
$p_3 = 0.67847$
$p_4 = 0.67846$
$p_5 = 0.67846$
$p_6 = 0.67846$
$p = 0.678$
$p = 0.678$ (to 3 decimal places)

20 **i** **a** $\int \frac{e^{2x}+6}{e^{2x}}\,dx$

$= \int \left(1+6e^{-2x}\right) dx$

$= x - 3e^{-2x} + c$

b $\int 3\cos^2 x\,dx$

$= 3\int \left(\frac{1}{2}+\frac{1}{2}\cos 2x\right) dx$

$= 3\left(\frac{1}{2}x+\frac{1}{4}\sin 2x\right) + c$

$= \frac{3}{2}x+\frac{3}{4}\sin 2x + c$

P2

ii $h = \frac{2-1}{2} = \frac{1}{2}$

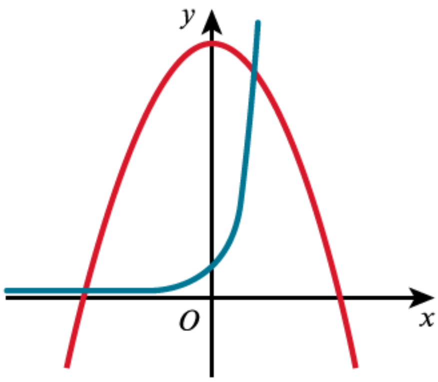
x	1	1.5	2
y	$\frac{6}{\ln(1+2)}$	$\frac{6}{\ln(1.5+2)}$	$\frac{6}{\ln(2+2)}$

$$\int_1^2 \frac{6}{\ln(x+2)}\,dx \approx \frac{h}{2}[y_0+y_2+2y_1]$$

$$= \frac{1}{4}\left[\frac{6}{\ln(1+2)}+\frac{6}{\ln(2+2)}+\frac{12}{\ln(1.5+2)}\right]$$

$= 4.84$ (to 2 decimal places)

21 **i** Sketching the graphs of $y = e^{2x}$ and $y = 14 - x^2$.



Note that the graphs intersect twice, so the given equation has two real roots.

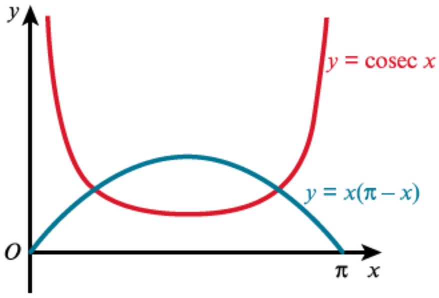
ii $f(x) = e^{2x} + x^2 - 14$
$f(1.2) = -1.53 < 0$
$f(1.3) = 1.15 > 0$
Change of sign indicates the presence of a root between 1.2 and 1.3.

iii $e^{2x} = 14 - x^2$
$2x = \ln(14 - x^2)$
$x = \frac{1}{2}\ln(14 - x^2)$

iv Using $x_{n+1} = \frac{1}{2}\ln\left(14 - x_n{}^2\right)$ with $x_0 = 1.25$:

$x_1 = 1.2604$
$x_2 = 1.2593$
$x_3 = 1.2594$
$x_4 = 1.2594$
$x = 1.26$ (to 2 decimal places)

22 **i**



Note that the graphs intersect twice, so the given equation has two real roots in the given interval.

ii

$$\begin{aligned}\operatorname{cosec} x &= x(\pi - x)\\ \frac{1}{\sin x} &= x(\pi - x)\\ 1 &= \pi x \sin x - x^2 \sin x\\ 1 + x^2 \sin x &= \pi x \sin x\\ x &= \frac{1 + x^2 \sin x}{\pi \sin x}\end{aligned}$$

iii **a** α is the smaller root and can be seen, from part **i**, to lie between 0 and $\frac{\pi}{2}$.

Using $x_{n+1} = \frac{1 + x_n{}^2 \sin x_n}{\pi \sin x_n}$ with $x_0 = 1$:

$x_1 = 0.6966$
$x_2 = 0.6506$
$x_3 = 0.6603$
$x_4 = 0.6577$
$x_5 = 0.6584$
$x_6 = 0.6582$
$\alpha = 0.66$ (to 2 decimal places)

b **Note that** $\sin(\pi - x) = \sin \pi$ by symmetry.

Similarly, $x(\pi - x)$ has the same value if x is replaced by $\pi - x$.

$\beta = \pi - \alpha = 2.48$ (to 2 decimal places)

23 **i**

$$\begin{aligned}x &= 2\ln(t+2)\\ \frac{dx}{dt} &= \frac{2}{t+2}\\ y &= t^3 + 2t + 3\\ \frac{dy}{dt} &= 3t^2 + 2\\ \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} = \left(3t^2 + 2\right) \times \frac{t+2}{2}\\ &= \frac{(3t^2+2)(t+2)}{2}\end{aligned}$$

At the origin, $x = y = 0$:

$$2\ln(t+2)=0$$
$$t+2=e^0=1$$
$$t=-1$$
$$\frac{dy}{dx}=\frac{(3+2)(-1+2)}{2}$$
$$=\frac{5}{2}$$

ii **a** At $P, t=p$

and $\frac{dy}{dx}=\frac{(3p^2+2)(p+2)}{2}=\frac{1}{2}$

$$3p^3+6p^2+2p+4=1$$
$$3p^3+6p^2+2p+3=0$$
$$3p^3+2p=-3-6p^2$$
$$p\left(3p^2+2\right)=-\left(3+6p^2\right)$$
$$p=-\frac{3+6p^2}{3p^2+2}$$
$$=-\frac{2\left(3p^2+2\right)-1}{3p^2+2}$$
$$=\frac{1}{3p^2+2}-2$$

For part **a** you could also use long division to find a quotient and remainder.

b Using $p_{n+1}=\frac{1}{3{p_n}^2+2}-2$ with $p_0=-2$:

$p_1=-1.92857$
$p_2=-1.92400$
$p_3=-1.92370$
$p_4=-1.92367$
$p_5=-1.92367$

$p=-1.92$ (to 2 decimal places)

$x=2\ln(-1.92367+2)=-5.15$ (to 2 decimal places)

$y=t^3+2t+3=-7.97$ (to 2 decimal places)

The point has coordinates $(-5.15, -7.97)$.

24 **i** $2\operatorname{cosec}2\theta\tan\theta$

$$\equiv\frac{2}{\sin 2\theta}\times\frac{\sin\theta}{\cos\theta}$$
$$\equiv\frac{2\sin\theta}{2\sin\theta\cos\theta\cos\theta}$$
$$\equiv\frac{1}{\cos^2\theta}$$
$$\equiv\sec^2\theta$$

ii **a** Using the result from part **i**:

$$\sec^2\theta=5$$
$$\cos^2\theta=\frac{1}{5}$$
$$\cos\theta=\frac{1}{\sqrt{5}}\text{ or }\cos\theta=-\frac{1}{\sqrt{5}}$$
$$\theta=1.11,\ 2.03$$

b Using the result from part **i**:

$2\operatorname{cosec}2\theta\tan\theta=\sec^2\theta$

Letting $\theta=2x$:

$2\operatorname{cosec}4x\tan 2x=\sec^2 2x$

$$\int_0^{\frac{\pi}{6}} \sec^2 2x dx$$
$$= \left[\frac{1}{2}\tan 2x\right]_0^{\frac{\pi}{6}}$$
$$= \frac{1}{2}\tan\frac{\pi}{3} - 0$$
$$= \frac{1}{2}\sqrt{3}$$

25 **i** You will need to use the identity $\cos 2x \equiv 2\cos^2 x - 1$ several times in this solution.

$$\cos 4\theta + 4\cos 2\theta$$
$$= 2\cos^2 2\theta - 1 + 4\cos 2\theta$$
$$= 2\left(2\cos^2\theta - 1\right)^2 - 1 + 4\left(2\cos^2\theta - 1\right)$$
$$= 2\left(4\cos^4\theta - 4\cos^2\theta + 1\right) - 1 + 8\cos^2\theta - 4$$
$$= 8\cos^4\theta - 8\cos^2\theta + 2 - 1 + 8\cos^2\theta - 4$$
$$= 8\cos^4\theta - 3$$

ii **a** Using the result from part **i**:

$$8\cos^4\theta - 3 = 1$$
$$\cos^4\theta = \frac{1}{2}$$
$$\cos\theta = \frac{1}{\sqrt[4]{2}} \text{ or } \cos\theta = -\frac{1}{\sqrt[4]{2}}$$
$\theta = \pm 0.572$ or $\theta = \pm 2.570$ (out of range)

$\theta = \pm 0.572$

b Using the result from part **i**:

$$8\cos^4\theta = 3 + \cos 4\theta + 4\cos 2\theta$$
$$\int_0^{\frac{\pi}{4}} \cos^4\theta \, d\theta$$
$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} (3 + \cos 4\theta + 4\cos 2\theta)\, d\theta$$
$$= \frac{1}{8}\left[3\theta + \frac{1}{4}\sin 4\theta + 2\sin 2\theta\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{8}\left(\frac{3\pi}{4} + 0 + 2\sin\frac{\pi}{2} - 0 - 0 - 0\right)$$
$$= \frac{3}{32}\pi + \frac{1}{4}$$

26 **i**

$$\cos 3x = \cos(2x + x)$$
$$= \cos 2x\cos x - \sin 2x\sin x$$
$$= (2\cos^2 x - 1)\cos x - 2\sin x\cos x\sin x$$
$$= 2\cos^3 x - \cos x - 2\sin^2 x\cos x$$
$$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$$
$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$
$$= 4\cos^3 x - 3\cos x$$

ii Using the result from part **i**:

$$\cos 3x = 4\cos^3 x - 3\cos x$$
$$2\cos^3 x - \frac{3}{2}\cos x = \frac{1}{2}\cos 3x$$
$$2\cos^3 x - \frac{3}{2}\cos x + \frac{1}{2}\cos x = \frac{1}{2}\cos 3x + \frac{1}{2}\cos x$$
$$2\cos^3 x - \cos x = \frac{1}{2}\cos 3x + \frac{1}{2}\cos x$$

$$\int_0^{\frac{\pi}{6}} \left(\frac{1}{2}\cos 3x + \frac{1}{2}\cos x\right) \mathrm{d}x$$
$$= \left[\frac{1}{6}\sin 3x + \frac{1}{2}\sin x\right]_0^{\frac{\pi}{6}}$$
$$= \frac{1}{6}\sin\frac{\pi}{2} + \frac{1}{2}\sin\frac{\pi}{6} - 0 - 0$$
$$= \frac{1}{6} + \frac{1}{2}\left(\frac{1}{2}\right)$$
$$= \frac{1}{6} + \frac{1}{4}$$
$$= \frac{5}{12}$$

27 **i** $\frac{\mathrm{d}y}{\mathrm{d}x} = -5\mathrm{e}^{-\frac{1}{2}x}\sin 4x + 40\mathrm{e}^{-\frac{1}{2}x}\cos 4x$

$= 5\mathrm{e}^{-\frac{1}{2}x}(8\cos 4x - \sin 4x)$

$\mathrm{e}^{-\frac{1}{2}x} > 0$

At all T:

$8\cos 4x - \sin 4x = 0$

$8\cos 4x = \sin 4x$

$\tan 4x = 8$

$4x = 1.44644133 + (n-1)\pi$, where $n = 1, 2, 3, 4, \ldots$

$x = \frac{1.44644133 + (n-1)\pi}{4}$

T_1 and T_2 occur at the two smallest values, with $n = 1$ and 2

$x = 0.362,\ 1.147$

ii $\frac{1.44644133 + (n-1)\pi}{4} > 25$

$1.44644133 + (n-1)\pi > 100$

$(n-1)\pi > 100 - 1.44644133$

$(n-1) > \frac{100 - 1.44644133}{\pi}$

$n > 32.4$

Smallest n satisfying this inequality is 33.

28 **i**

Remember the quotient rule: If $y = \frac{u}{v}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$.

$y = \frac{3x^2}{x^2+4}$

Using the quotient rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2+4)(6x) - 3x^2(2x)}{(x^2+4)^2}$$
$$= \frac{6x^3 + 24x - 6x^3}{(x^2+4)^2}$$
$$= \frac{24x}{(x^2+4)^2}$$

Using the fact that the gradient is $\frac{1}{2}$ at $x = p$:

$$\frac{24p}{(p^2+4)^2} = \frac{1}{2}$$
$$48p = (p^2+4)^2$$
$$48p = p^4 + 8p^2 + 16$$
$$p^4 + 8p^2 - 48p + 16 = 0$$
$$p^2(p^2+8) = 48p - 16$$
$$p^2 = \frac{48p-16}{p^2+8}$$
$$p = \sqrt{\frac{48p-16}{p^2+8}}$$

ii $\mathrm{f}(p) = \sqrt{\frac{48p-16}{p^2+8}} - p$

$\mathrm{f}(2) = \sqrt{\frac{80}{12}} - 2 = 0.582 > 0$

$\mathrm{f}(3) = \sqrt{\frac{128}{17}} - 3 = -0.256 < 0$

Change of sign indicates the presence of a root between 2 and 3.

So $2 < p < 3$

iii Using $p_{n+1} = \sqrt{\frac{48p_n - 16}{p_n^2+8}}$ with $p_0 = 2.5$:

$p_1 = 2.70153$
$p_2 = 2.72589$
$p_3 = 2.72811$
$p_4 = 2.72831$
$p_5 = 2.72832$
$p_6 = 2.72833$
$p = 2.728$ (to 4 significant figures)

29 **i** $y = \sec\theta$

$$= \frac{1}{\cos\theta}$$
$$= (\cos\theta)^{-1}$$
$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -(\cos\theta)^{-2}(-\sin\theta)$$
$$= \frac{\sin\theta}{\cos^2\theta}$$
$$= \frac{\sin\theta}{\cos\theta} \times \frac{1}{\cos\theta}$$
$$= \tan\theta\sec\theta$$

ii $\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2} = \frac{\mathrm{d}}{\mathrm{d}\theta}(\tan\theta\sec\theta)$

$$= \sec^2\theta\sec\theta + \tan\theta\tan\theta\sec\theta$$
$$= \sec^3\theta + \tan^2\theta\sec\theta$$
$$= \sec^3\theta + (\sec^2\theta - 1)\sec\theta$$
$$= \sec^3\theta + \sec^3\theta - \sec\theta$$
$$= 2\sec^3\theta - \sec\theta$$

So $a = 2$ and $b = -1$.

iii $1 + \tan^2\theta = \sec^2\theta$

$$
\begin{aligned}
&\int_0^{\frac{\pi}{4}} \left(1 + \tan^2\theta - 3\sec\theta\tan\theta\right) \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \left(\sec^2\theta - 3\sec\theta\tan\theta\right) \mathrm{d}\theta \\
&= [\tan\theta - 3\sec\theta]_0^{\frac{\pi}{4}} \\
&= \tan\frac{\pi}{4} - \frac{3}{\cos\frac{\pi}{4}} - 0 + \frac{3}{\cos 0} \\
&= 1 - \frac{3}{\left(\frac{1}{\sqrt{2}}\right)} + 3 \\
&= 4 - 3\sqrt{2}
\end{aligned}
$$

Chapter 7

Further algebra

P3 This chapter is for Pure Mathematics 3 students only.

EXERCISE 7A

1 d

$$
\begin{array}{r}
x+2 \\
x^2+2x+3\overline{\big)\,x^3+4x^2+3x-1} \\
\underline{x^3+2x^2+3x} \\
2x^2+0x-1 \\
\underline{2x^2+4x+6} \\
-4x-7
\end{array}
$$

$$
\begin{aligned}
\frac{x^3+4x^2+3x-1}{x^2+2x+3} &= x+2+\frac{-4x-7}{x^2+2x+3} \\
&= x+2-\frac{4x+7}{x^2+2x+3}
\end{aligned}
$$

f

$$
\begin{array}{r}
x^2+1 \\
x^2+0x+1\overline{\big)\,x^4+0x^3+2x^2+0x-5} \\
\underline{x^4+0x^3+x^2} \\
x^2+0x-5 \\
\underline{x^2+0x+1} \\
-6
\end{array}
$$

$$\frac{x^4+0x^3+2x^2+0x-5}{x^2+1} = x^2+1-\frac{6}{x^2+1}$$

It is crucial, again, to remember that you need to include ‘missing’ terms using zero coefficients. If you don’t, the terms won’t line up and it will be difficult to keep track during the subtraction stages.

2

$$
\begin{array}{r}
x^2+4x+12 \\
x-3\overline{\big)\,x^3+x^2+0x-7} \\
\underline{x^3-3x^2} \\
4x^2+0x \\
\underline{4x^2-12x} \\
12x-7 \\
\underline{12x-36} \\
29
\end{array}
$$

$$\frac{x^3+x^2-7}{x-3} = x^2+4x+12+\frac{29}{x-3}$$

$A = 1,\ B = 4,\ C = 12,\ D = 29.$

3

$$
\begin{array}{r}
x^3-x^2+6x-6 \\
x+1\overline{\big)\,x^4+0x^3+5x^2+0x-1}
\end{array}
$$

$$\begin{array}{l}
\underline{x^4 + x^3} \\
\quad - x^3 + 5x^2 \\
\quad - \underline{x^3 - x^2} \\
\qquad\quad 6x^2 + 0x \\
\qquad\quad \underline{6x^2 + 6x} \\
\qquad\qquad -6x - 1 \\
\qquad\qquad \underline{-6x - 6} \\
\qquad\qquad\qquad\quad 5
\end{array}$$

$$\frac{x^4 + 5x^2 - 1}{x + 1} = x^3 - x^2 + 6x - 6 + \frac{5}{x + 1}$$

$A = 1,\ B = -1,\ C = 6,\ D = -6,\ E = 5.$

4

$$\begin{array}{r}
2x + 3 \\
x^3 + 0x^2 + 2x \overline{\big)\, 2x^4 + 3x^3 + 4x^2 + 5x + 6} \\
\underline{2x^4 + 0x^3 + 4x^2} \qquad\qquad\quad \\
3x^3 + 0x^2 + 5x \quad\;\; \\
\underline{3x^3 + 0x^2 + 6x} \quad\;\; \\
- x + 6
\end{array}$$

$$\frac{2x^4 + 3x^3 + 4x^2 + 5x + 6}{x^3 + 2x} = 2x + 3 + \frac{6 - x}{x^3 + 2x}$$

$A = 2,\ B = 3,\ C = -1,\ D = 6.$

EXERCISE 7B

1 **a**

Two different methods are shown for the two parts covered in Question **1**. worked solution **1 a** shows substituting particular values of x. worked solution **1 e** shows comparing coefficients. Unless instructed otherwise, you should use the method that you are most confident with. You can always use the other method as a check.

$$\frac{6x-2}{(x-2)(x+3)}=\frac{A}{x-2}+\frac{B}{x+3}$$
$$6x-2=A(x+3)+B(x-2)$$

Letting $x=-3$:

$$-20=0-5B$$
$$B=4$$

Letting $x=2$:

$$10=5A+0$$
$$A=2$$
$$\frac{6x-2}{(x-2)(x+3)}=\frac{2}{x-2}+\frac{4}{x+3}$$

e $$\frac{6x^2+5x-2}{x(x-1)(2x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{2x+1}$$
$$6x^2+5x-2=A(x-1)(2x+1)+Bx(2x+1)+Cx(x-1)$$

Calculating coefficients of x^2:

$6=2A+2B+C$ .... [1]

Equating coefficients of x:

$5=-A+B-C$ .... [2]

Equating constant term:

$$-2=-A$$
$$A=2$$

Substituting $A=2$ into [1]:

$6=4+2B+C$

$2B+C=2$ .... [3]

Substituting $A=2$ into [2]:

$5=-2+B-C$ ..... [4]

$B-C=7$

[3] + [4]:

$$3B=9$$
$$B=3$$

Substituting $B=3$ into [4]:

$$3-C=7$$
$$C=-4$$
$$\frac{6x^2+5x-2}{x(x-1)(2x+1)}=\frac{2}{x}+\frac{3}{x-1}-\frac{4}{2x+1}$$

2 **b** $$\frac{11x^2+14x+5}{(2x+1)(x+1)^2}=\frac{A}{2x+1}+\frac{B}{(x+1)}+\frac{C}{(x+1)^2}$$
$$11x^2+14x+5=A(x+1)^2+B(2x+1)(x+1)+C(2x+1)$$

Letting $x=-1$:

$$11-14+5=0+0-C$$
$$C=-2$$

Letting $x=-\frac{1}{2}$:

$$\frac{11}{4} - 7 + 5 = \frac{1}{4}A + 0 + 0$$
$$\frac{11}{4} - \frac{8}{4} = \frac{1}{4}A$$
$$A = 3$$

Letting $x = 0$:

$$5 = A + B + C$$
$$5 = 3 + B - 2$$
$$B = 4$$
$$\frac{11x^2 + 14x + 5}{(2x+1)(x+1)^2} = \frac{3}{2x+1} + \frac{4}{(x+1)} - \frac{2}{(x+1)^2}$$

Notice that putting $x = 0$ is the same as equating constant terms. So this method is a combination of using particular values of x and equating coefficients.

e $$\frac{3}{(x+2)(x-2)^2} = \frac{A}{x+2} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$
$$3 = A(x-2)^2 + B(x+2)(x-2) + C(x+2)$$

Letting $x = 2$:

$$3 = 0 + 0 + 4C$$
$$C = \frac{3}{4}$$

Letting $x = -2$:

$$3 = 16A + 0 + 0$$
$$A = \frac{3}{16}$$

Letting $x = 0$:

$$3 = 4A - 4B + 2C$$
$$3 = \frac{3}{4} - 4B + \frac{3}{2}$$
$$12 = 3 - 16B + 6$$
$$16B = -3$$
$$B = -\frac{3}{16}$$
$$\frac{3}{(x+2)(x-2)^2} = \frac{3}{16\,(x+2)} - \frac{3}{16(x-2)} + \frac{3}{4(x-2)^2}$$

3 b $$\frac{3x^2 + 4x + 17}{(2x+1)(x^2+5)} = \frac{A}{(2x+1)} + \frac{Bx + C}{x^2 + 5}$$
$$3x^2 + 4x + 17 = A(x^2+5) + (Bx + C)(2x+1)$$

Letting $x = -\frac{1}{2}$:

$$\frac{3}{4} - 2 + 17 = \frac{21}{4}A + 0$$
$$21A = 3 - 8 + 68$$
$$A = \frac{63}{21} = 3$$

Letting $x = 0$:

$$17 = 5A + C$$
$$17 = 15 + C$$
$$C = 2$$

Equating coefficients of x^2:

$$3 = A + 2B$$
$$3 = 3 + 2B$$
$$B = 0$$
$$\frac{3x^2 + 4x + 17}{(2x+1)(x^2+5)} = \frac{3}{(2x+1)} + \frac{2}{x^2+5}$$

c $\dfrac{2x^2-6x-9}{(3x+5)(2x^2+1)}=\dfrac{A}{3x+5}+\dfrac{Bx+C}{2x^2+1}$

$$2x^2-6x-9=A(2x^2+1)+(Bx+C)(3x+5)$$

Letting $x=-\dfrac{5}{3}$:

$$\frac{50}{9}+10-9=\frac{59}{9}A+0$$
$$59A=59$$
$$A=1$$

Letting $x=0$:

$$-9=A+5C$$
$$5C=-9-1=-10$$
$$C=-2$$

Equating coefficients of x^2:

$$2=2A+3B$$
$$2=2+3B$$
$$B=0$$
$$\frac{2x^2-6x-9}{(3x+5)(2x^2+1)}=\frac{1}{3x+5}-\frac{2}{2x^2+1}$$

4 a Remember to expand the brackets in the denominator when dividing.

$(x-1)(x+2)=x^2+x-2$

$$\begin{array}{r} 2 \\ x^2+x-2\overline{\big)\,2x^2+3x+4} \\ \underline{2x^2+2x-4} \\ x+8 \end{array}$$

$$\frac{2x^2+3x+4}{(x-1)(x+2)}=2+\frac{x+8}{(x-1)(x+2)}$$

Splitting the proper fraction $\dfrac{x+8}{(x-1)(x+2)}$ using partial fractions:

$$\frac{x+8}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}$$
$$x+8=A(x+2)+B(x-1)$$

Letting $x=-2$:

$$6=0-3B$$
$$B=-2$$

Letting $x=1$:

$$9=3A+0$$
$$A=3$$

So $\dfrac{2x^2+3x+4}{(x-1)(x+2)}=2+\dfrac{3}{x-1}-\dfrac{2}{x+2}$

b

$$\begin{array}{r} 1 \\ x^2+0x-4\overline{\big)\,x^2+0x+3} \\ \underline{x^2+0x-4} \\ 7 \end{array}$$

$$\frac{x^2+3}{x^2-4}=1+\frac{7}{x^2-4}$$

Splitting the proper fraction $\dfrac{7}{x^2-4}$ using partial fractions:

$$\frac{7}{x^2-4}=\frac{7}{(x-2)(x+2)}=\frac{A}{x-2}+\frac{B}{x+2}$$
$$7=A(x+2)+B(x-2)$$

Letting $x=2$:

$7 = 4A + 0$

$A = \frac{7}{4}$

Letting $x = -2$:

$7 = 0 - 4B$

$B = -\frac{7}{4}$

So $\frac{x^2+3}{x^2-4} = 1 + \frac{7}{4(x-2)} - \frac{7}{4(x+2)}$

5 $$\begin{array}{r} 2 \\ 2x^3 - x^2 \overline{\smash{)}4x^3 - 9x^2 + 11x - 4} \\ \underline{4x^3 - 2x^2} \qquad\qquad \\ -7x^2 + 11x - 4 \end{array}$$

$\frac{4x^3 - 9x^2 + 11x - 4}{x^2(2x-1)} = 2 + \frac{-7x^2 + 11x - 4}{x^2(2x-1)}$

Splitting the proper fraction $\frac{-7x^2 + 11x - 4}{x^2(2x-1)}$ using partial fractions:

$\frac{-7x^2 + 11x - 4}{x^2(2x-1)} = \frac{p}{x} + \frac{q}{x^2} + \frac{r}{2x-1}$

$-7x^2 + 11x - 4 = px(2x-1) + q(2x-1) + rx^2$

Letting $x = 0$:

$-4 = 0 - q + 0$

$q = 4$

Letting $x = \frac{1}{2}$:

$-\frac{7}{4} + \frac{11}{2} - 4 = 0 + 0 + \frac{1}{4}r$

$r = -7 + 22 - 16 = -1$

Equating coefficients of x^2:

$-7 = 2p + r$

$-7 = 2p - 1$

$2p = -6$

$p = -3$

So $\frac{4x^3 - 9x^2 + 11x - 4}{x^2(2x-1)} = 2 - \frac{3}{x} + \frac{4}{x^2} - \frac{1}{2x-1}$

$A = 2,\ B = -3,\ C = 4,\ D = -1.$

6 **a** Let $f(x) = 2x^3 - 3x^2 - 3x + 2$

$f(-1) = -2 - 3 + 3 + 2 = 0$

So $x + 1$ is a key factor of $f(x)$ by the factor theorem.

Dividing $2x^3 - 3x^2 - 3x + 2$ by $x + 1$ using long division:

$$\begin{array}{r} 2x^2 - 5x + 2 \\ x+1 \overline{\smash{)}2x^3 - 3x^2 - 3x + 2} \\ \underline{2x^3 + 2x^2} \qquad\qquad\quad \\ -5x^2 - 3x \qquad \\ \underline{-5x^2 - 5x} \qquad \\ 2x + 2 \\ \underline{2x + 2} \\ 0 \end{array}$$

$2x^3 - 3x^2 - 3x + 2 = (x+1)(2x^2 - 5x + 2)$

$= (x+1)(2x-1)(x-2)$

b $\frac{x^2 - 13x - 5}{2x^3 - 3x^2 - 3x + 2}$

$$= \frac{x^2 - 13x - 5}{(x+1)(2x-1)(x-2)}$$
$$= \frac{A}{x+1} + \frac{B}{2x-1} + \frac{C}{x-2}$$

$x^2 - 13x - 5 = A(2x-1)(x-2) + B(x+1)(x-2) + C(x+1)(2x-1)$

Letting $x = 2$:

$$\begin{aligned} 4 - 26 - 5 &= 0 + 0 + 9C \\ 9C &= -27 \\ C &= -3 \end{aligned}$$

Letting $x = \frac{1}{2}$:

$$\begin{aligned} \frac{1}{4} - \frac{13}{2} - 5 &= 0 - \frac{9}{4}B + 0 \\ -9B &= 1 - 26 - 20 \\ -9B &= -45 \\ B &= 5 \end{aligned}$$

Letting $x = -1$:

$$\begin{aligned} 1 + 13 - 5 &= 9A + 0 + 0 \\ 9A &= 9 \\ A &= 1 \end{aligned}$$

$$\frac{x^2 - 13x - 5}{2x^3 - 3x^2 - 3x + 2} = \frac{1}{x+1} + \frac{5}{2x-1} - \frac{3}{x-2}$$

7 **a** Let $f(x) = 2x^3 - 11x^2 + 12x + 9$

$f(3) = 54 - 99 + 36 + 9 = 0$

So $x - 3$ is a factor of $f(x)$ by the factor theorem.

Dividing $2x^3 - 11x^2 + 12x + 9$ by $x - 3$ using long division:

$$\begin{array}{r} 2x^2 - 5x - 3 \\ x - 3\,\overline{)\,2x^3 - 11x^2 + 12x + 9} \\ \underline{2x^3 - 6x^2} \qquad\qquad \\ -5x^2 + 12x \qquad \\ \underline{-5x^2 + 15x} \qquad \\ -3x + 9 \\ \underline{-3x + 9} \\ 0 \end{array}$$

$$\begin{aligned} 2x^3 - 11x^2 + 12x + 9 &= (x-3)(2x^2 - 5x - 3) \\ &= (x-3)(2x+1)(x-3) \\ &= (2x+1)(x-3)^2 \end{aligned}$$

Make sure you check for repeated factors. If you don't spot them, the method used in part **b** won't work.

b $$\frac{24 - x}{2x^3 - 11x^2 + 12x + 9} = \frac{24 - x}{(2x+1)(x-3)^2}$$

$$\frac{24 - x}{(2x+1)(x-3)^2} = \frac{A}{2x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

$$24 - x = A(x-3)^2 + B(2x+1)(x-3) + C(2x+1)$$

Letting $x = 3$:

$$\begin{aligned} 21 &= 0 + 0 + 7C \\ C &= 3 \end{aligned}$$

Letting $x = -\frac{1}{2}$:

$$\begin{aligned} \frac{49}{2} &= \frac{49}{4}A + 0 + 0 \\ A &= 2 \end{aligned}$$

Equating coefficients of x^2:

$$0 = A + 2B$$
$$0 = 2 + 2B$$
$$B = -1$$

$$\frac{24 - x}{2x^3 - 11x^2 + 12x + 9} = \frac{2}{2x + 1} - \frac{1}{x - 3} + \frac{3}{(x - 3)^2}$$

8 **a** $p(x) = 2x^3 + 5x^2 + ax + b$

By the factor theorem:

$$p\left(-\frac{1}{2}\right) = 0$$
$$-\frac{1}{4} + \frac{5}{4} - \frac{1}{2}a + b = 0$$
$$-1 + 5 - 2a + 4b = 0$$
$$2a - 4b = 4$$
$$a - 2b = 2 \text{ .... } [1]$$

By the remainder theorem:

$$p(-2) = 9$$
$$-16 + 20 - 2a + b = 9$$
$$2a - b = -5 \text{ .... } [2]$$

$2 \times$ [2] :

$$4a - 2b = -10 \text{ .... } [3]$$

[3]– [1] :

$$3a = -12$$
$$a = -4$$

Substituting $a = -4$ into [1]:

$$-4 - 2b = 2$$
$$2b = -6$$
$$b = -3$$

b Dividing $p(x)$ by $2x + 1$ using long division:

$$\begin{array}{r} x^2 + 2x - 3 \\ 2x + 1 \overline{\smash{)}\, 2x^3 + 5x^2 - 4x - 3} \\ \underline{2x^3 + x^2} \qquad\qquad \\ 4x^2 - 4x \qquad \\ \underline{4x^2 + 2x} \qquad \\ -6x - 3 \\ \underline{-6x - 3} \\ 0 \end{array}$$

$$2x^3 + 5x^2 - 4x - 3 = (2x + 1)(x^2 + 2x - 3)$$
$$= (2x + 1)(x + 3)(x - 1)$$

c $$\frac{120}{p(x)} = \frac{120}{(2x + 1)(x + 3)(x - 1)}$$
$$= \frac{A}{2x + 1} + \frac{B}{x + 3} + \frac{C}{x - 1}$$
$$120 = A(x + 3)(x - 1) + B(2x + 1)(x - 1) + C(2x + 1)(x + 3)$$

Letting $x = 1$:

$$120 = 0 + 0 + 12C$$
$$C = 10$$

Letting $x =-3$:

$$120 = 0 + 20B + 0$$
$$B = 6$$

Letting $x = -\frac{1}{2}$:

$$120=-\frac{15}{4}A+0+0$$
$$A=-32$$
$$\frac{120}{\mathrm{p}(x)}=-\frac{32}{2x+1}+\frac{6}{x+3}+\frac{10}{x-1}$$

9 **a** $$\frac{2}{x(x+2)}=\frac{A}{x}+\frac{B}{x+2}$$
$$2=A(x+2)+Bx$$

Letting $x=0$:

$$2=2A+0$$
$$A=1$$

Letting $x=-2$:

$$2=0-2B$$
$$B=-1$$
$$\frac{2}{x(x+2)}=\frac{1}{x}-\frac{1}{x+2}$$

b Using the result from part **a**:

$$\frac{2}{1\times 3}+\frac{2}{2\times 4}+\frac{2}{3\times 5}+\frac{2}{4\times 6}+\frac{2}{5\times 7}+\ldots$$
$$=\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\ldots$$
$$+\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right)$$
$$=1+\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\frac{1}{5}-\frac{1}{5}+\ldots+\frac{1}{n}-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}$$
$$=1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}=\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}$$

c Sum to infinity

$$\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\to\frac{3}{2}-0-0=\frac{3}{2}\text{ as } n\to\infty$$

10 Consider

$$\frac{1}{n(n+1)(n+2)}=\frac{A}{n}+\frac{B}{n+1}+\frac{C}{n+2}$$
$$1=A(n+1)(n+2)+Bn(n+2)+Cn(n+1)$$

Letting $n=-1$:

$$1=0-B+0$$
$$B=-1$$

Letting $n=0$:

$$1=2A+0+0$$
$$A=\frac{1}{2}$$

Letting $n=-2$:

$$1=0+0+2C$$
$$C=\frac{1}{2}$$
$$\frac{1}{n(n+1)(n+2)}=\frac{1}{2n}-\frac{1}{n+1}+\frac{1}{2(n+2)}$$
$$\frac{1}{1\times 2\times 3}+\frac{1}{2\times 3\times 4}+\frac{1}{3\times 4\times 5}+\ldots$$
$$=\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right)+\left(\frac{1}{4}-\frac{1}{3}+\frac{1}{8}\right)+\ldots\left(\frac{1}{2(n-2)}-\frac{1}{n-2}+\frac{1}{2n}\right)$$
$$+\left(\frac{1}{2(n-1)}-\frac{1}{n}+\frac{1}{2(n+1)}\right)+\left(\frac{1}{2n}-\frac{1}{n+1}+\frac{1}{2(n+2)}\right)$$
$$=\frac{1}{4}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}$$

Telescoping with $S_\infty = \frac{1}{4} - 0 + 0 = \frac{1}{4}$

EXERCISE 7C

1 **a** $(1+x)^{-2} = 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \ldots$

$= 1 - 2x + 3x^3 - 4x^3 + \ldots$

Expansion is valid for $|x| < 1$.

e $\sqrt{1+2x} = (1+2x)^{\frac{1}{2}}$

$= 1 + \left(\frac{1}{2}\right)(2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(2x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(2x)^3 + \ldots$

$= 1 + x - \frac{4}{8}x^2 + \frac{3 \times 8}{8 \times 6}x^3 + \ldots$

$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \ldots$

Expansion is valid for $|2x| < 1$

and so for $|x| < \frac{1}{2}$.

2 **a** $(1+x^2)^{-3}$

$= 1 + (-3)(x^2) + \frac{(-3)(-4)}{2!}(x^2)^2 + \frac{(-3)(-4)(-5)}{3!}(x^2)^3 + \ldots$

$= 1 - 3x^2 + 6x^4 + \ldots$

Expansion is valid for $|x^4| < 1$

and so for $|x| < 1$.

b $\sqrt[3]{1-2x^2} = (1-2x^2)^{\frac{1}{3}}$

$= 1 + \left(\frac{1}{3}\right)(-2x^2) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(-2x^2)^2 + \ldots$

$= 1 - \frac{2}{3}x^2 - \frac{2 \times 4}{9 \times 2}x^4 + \ldots$

$= 1 - \frac{2}{3}x^2 - \frac{4}{9}x^4 + \ldots$

Expansion is valid for $|-2x^2| < 1$

$|x^2| < \frac{1}{2}$

$|x| < \sqrt{\frac{1}{2}}$

c $\left(\sqrt{1-4x^2}\right)^3 = (1-4x^2)^{\frac{3}{2}}$

$= 1 + \left(\frac{3}{2}\right)(-4x^2) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}(4x^2)^2 + \ldots$

$= 1 - 6x^2 + \frac{3 \times 16}{4 \times 2}x^4 + \ldots$

$= 1 - 6x^2 + 6x^4 - \ldots$

Expansion is valid for $|-4x^2| < 1$

$|x^2| < \frac{1}{4}$

$|x| < \frac{1}{2}$

3 $\frac{2+3x}{\sqrt{1-5x^2}} = (2+3x)(1-5x^2)^{-\frac{1}{2}}$

Considering $(1-5x^2)^{-\frac{1}{2}}$:

$$(1-5x^2)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-5x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-5x^2)^2 + \dots$$
$$= 1 + \frac{5}{2}x^2 + \frac{3\times 25}{4\times 2}x^4 + \dots$$
$$= 1 + \frac{5}{2}x^2 + \frac{75}{8}x^4 + \dots$$
$$(2+3x)(1-5x^2)^{-\frac{1}{2}} = (2+3x)\left(1 + \frac{5}{2}x^2 + \frac{75}{8}x^4 + \dots\right)$$
$$= 2 + 3x + 2\left(\frac{5}{2}\right)x^2 + 3\left(\frac{5}{2}\right)x^3 + 2\left(\frac{75}{8}\right)x^4 + 3\left(\frac{75}{8}\right)x^5 + \dots$$
$$= 2 + 3x + 5x^2 + \frac{15}{2}x^3 + \frac{75}{4}x^4 + \dots$$

4 **a** $(3x-1)^{-2} = \dfrac{1}{(3x-1)^2}$

$$= \frac{1}{(-1)^2(1-3x)^2}$$
$$= (1-3x)^{-2}$$

This is now in the form $(1+X)^n$, which is possible to expand.

> The expression in part **a** has worked because the power is even. When you take an even power of a negative number you get a positive answer.

b Note that $\sqrt{2x-1}$

$= \sqrt{-(1-2x)}$

But you cannot take the square root of the -1 to obtain a real number and, therefore, cannot write the expression in the form $(1+X)^n$. This function cannot be expanded using the binomial expansion.

5 $(2x-1)^{-3} = [-(1-2x)]^{-3}$

$$= (-1)^{-3}(1-2x)^{-3}$$
$$= -(1-2x)^{-3}$$
$$= -\left\{1 + (-3)(-2x) + \frac{(-3)(-4)}{2!}(-2x)^2 + \frac{(-3)(-4)(-5)}{3!}(-2x)^3 + \dots\right\}$$
$$= -1 - 6x - 24x^2 - 80x^3 - \dots$$

6 $(1+x)^n$ can only be expanded if $|x| < 1$

But, if $x = 3$ then $|x| = 3 > 1$ and so the expansion is not valid.

7 $(1-3x)^{-4}$

$$= 1 + (-4)(-3x) + \frac{(-4)(-5)}{2!}(-3x)^2 + \dots$$
$$= 1 + 12x + 90x^2 + \dots$$
$$(1+2x)^{\frac{3}{2}} = 1 + \left(\frac{3}{2}\right)(2x) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}(2x)^2 + \dots$$
$$= 1 + 3x + \frac{3\times 4}{4\times 2}x^2 + \dots$$
$$= 1 + 3x + \frac{3}{2}x^2 + \dots$$

So

$$(1-3x)^{-4} - (1+2x)^{\frac{3}{2}}$$
$$= 1 + 12x + 90x^2 + \dots - \left(1 + 3x + \frac{3}{2}x^2 + \dots\right)$$
$$= 1 - 1 + 12x - 3x + \left(90 - \frac{3}{2}\right)x^2 + \dots$$
$$= 9x + \frac{177}{2}x^2 + \dots$$

So $k = \frac{177}{2}$

Question **7** mentions 'small values' of x. If x is small, then you can usually say that powers of x that are 3 or greater are so close to zero that you can neglect them.

8 $\frac{1}{1-x} = (1-x)^{-1}$

$= 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}x^2 + \dots$

$= 1 + x + x^2 + \dots$

$\frac{1}{1+2x} = (1+2x)^{-1}$

$= 1 + (-1)(2x) + \frac{(-1)(-2)}{2!}(2x)^2 + \dots$

$= 1 - 2x + 4x^2 + \dots$

So $\frac{a}{1-x} + \frac{b}{1+2x} = a(1 + x + x^2 + \dots) + b(1 - 2x + 4x^2 + \dots)$

$= -3 + 12x$

Equating constants: $-3 = a + b$ .... [1]

Equating coefficients of x: $12 = a - 2b$ ...... [2]

[1]– [2]:

$3b = -15$

$b = -5$

Substituting $b = -5$ into [1]:

$-3 = a - 5$

$a = 2$

9 **a** $(1 + ax)^{-3}$

$= 1 + (-3)(ax) + \frac{(-3)(-4)}{2!}(ax)^2 + \dots$5b

$= 1 - 3ax + 6a^2x^2 + \dots$

Using the fact that the coefficients of x^2 and x are equal:

$6a^2 = -3a$

$6a^2 + 3a = 0$

$3a(2a + 1) = 0$

$a = 0$ or $a = -\frac{1}{2}$

But $a < 0$

so $a = -\frac{1}{2}$

b

Expand using the letter a first, then substitute the correct value for a at the end. If you substitute early in the working, then the algebra gets much harder!

$$(1 + ax)^{-3} = 1 + (-3)(ax) + \frac{(-3)(-4)}{2!}(ax)^2 + \frac{(-3)(-4)(-5)}{3!}(ax)^3 + \frac{(-3)(-4)(-5)(-6)}{4!}(ax)^4 + \dots$$

$$= 1 - 3ax + 6a^2x^2 - 10a^3x^3 + 15a^4x^4 - \dots$$

$$= 1 + \frac{3}{2}x + \frac{6}{4}x^2 + \frac{10}{8}x^3 + \frac{15}{16}x^4 + \dots$$

$$= 1 + \frac{3}{2}x + \frac{3}{2}x^2 + \frac{5}{4}x^3 + \frac{15}{16}x^4 + \dots$$

10 $(1+ax)^{\frac{2}{3}} = 1 + \left(\frac{2}{3}\right)(ax) + \frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{2!}(ax)^2 + \ldots$

$= 1 + \frac{2}{3}ax - \frac{1}{9}a^2x^2 + \ldots$

So $(3-2x)(1+ax)^{\frac{2}{3}}$

$= (3-2x)\left(1 + \frac{2}{3}ax - \frac{1}{9}a^2x^2 + \ldots\right)$

Using the fact that the coefficient of x^2 is -15:

$$3\left(-\frac{1}{9}a^2\right) - 2\left(\frac{2}{3}a\right) = -\frac{1}{3}a^2 - \frac{4}{3}a = -15$$

$$a^2 + 4a - 45 = 0$$

$$(a+9)(a-5) = 0$$

$a = -9$ or $a = 5$

11 **a** $(1+ax)^n = 1 + n(ax) + \frac{n(n-1)}{2!}(ax)^2 + \ldots$

$= 1 + anx + \frac{1}{2}n(n-1)a^2x^2 + \ldots$

$= 1 - 24x + 384x^2 + \ldots$

Equating coefficients of x: $an = -24$ .... [1]

Equating coefficients of x^2:

$\frac{1}{2}n(n-1)a^2 = 384$

$n(n-1)a^2 = 768$ .... [2]

From [1]: $a = -\frac{24}{n}$

Substituting into [2]:

$$n(n-1)\left(-\frac{24}{n}\right)^2 = 768$$

$$\frac{576n(n-1)}{n^2} = 768$$

$$576(n-1) = 768n$$

$$576n - 576 = 768n$$

$$192n = -576$$

$$n = -\frac{1152}{192} = -3$$

Substituting into [1]:

$-3a = -24$

$a = 8$

b Term in x^3 :

$\frac{n(n-1)(n-2)}{3!}(ax)^3$

$= \frac{(-3)(-4)(-5)}{3!}(8x)^3$

$= -5120x^3$

Note that part **b** asks for the entire term and not just the coefficient. This means that you also need to include the x^3 in your answer.

EXERCISE 7D

1 **b**
$$(5-2x)^{-1} = \left[5\left(1-\frac{2}{5}x\right)\right]^{-1}$$
$$= \frac{1}{5}\left[\left(1-\frac{2}{5}x\right)^{-1}\right]$$
$$= \frac{1}{5}\left[1+(-1)\left(-\frac{2}{5}x\right)+\frac{(-1)(-2)}{2!}\left(-\frac{2}{5}x\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(-\frac{2}{5}x\right)^3+\ldots\right]$$
$$= \frac{1}{5}+\frac{2}{25}x+\frac{4}{125}x^2+\frac{8}{625}x^3+\ldots$$

Valid for $\left|-\frac{2}{5}x\right|<1$

and so for $|x|<\frac{5}{2}$.

f

When taking the product of a function that needs to be expanded and another polynomial that is already in expanded form, deal with the expansion first, on its own.

$$\frac{1+2x}{(2x-5)^3} = (1+2x)(2x-5)^{-3}$$
$$(2x-5)^{-3} = \left[-5\left(1-\frac{2}{5}x\right)\right]^{-3}$$
$$= (-5)^{-3}\left(1-\frac{2}{5}x\right)^{-3}$$
$$= -\frac{1}{125}\left\{1+(-3)\left(-\frac{2}{5}x\right)+\frac{(-3)(-4)}{2!}\left(-\frac{2}{5}x\right)^2 + \frac{(-3)(-4)(-5)}{3!}\left(-\frac{2}{5}x\right)^3+\ldots\right\}$$
$$= -\frac{1}{125}-\frac{6}{625}x-\frac{24}{3125}x^2-\frac{16}{3125}x^3-\ldots$$

So $(1+2x)(2x-5)^{-3}$

$$= (1+2x)\left(-\frac{1}{125}-\frac{6}{625}x-\frac{24}{3125}x^2-\frac{16}{3125}x^3-\ldots\right)$$
$$= -\frac{1}{125}+\left(-\frac{2}{125}-\frac{6}{625}\right)x+\left(-\frac{24}{3125}-\frac{12}{625}\right)x^2+\left(-\frac{16}{3125}-\frac{48}{3125}\right)x^3+\ldots$$
$$= -\frac{1}{125}-\frac{16}{125}x-\frac{84}{3125}x^2-\frac{64}{3125}x^3+\ldots$$

Valid for $\left|-\frac{2}{5}x\right|<1$

and so for $|x|<\frac{5}{2}$.

2 **a**
$$(2+x^2)^{-2} = \left[2\left(1+\frac{x^2}{2}\right)\right]^{-2}$$
$$= 2^{-2}\left\{\left(1+\frac{x^2}{2}\right)^{-2}\right\}$$
$$= \frac{1}{4}\left\{1+(-2)\left(\frac{x^2}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x^2}{2}\right)^2+\ldots\right\}$$
$$= \frac{1}{4}-\frac{1}{4}x^2+\frac{3}{16}x^4+\ldots$$

Valid for $\left|\frac{x^2}{2}\right| < 1$

and so for $|x| < \sqrt{2}$.

b $$\begin{aligned}(8-3x^2)^{\frac{1}{3}} &= \left[8\left(1-\frac{3}{8}x^2\right)\right]^{\frac{1}{3}} \\ &= 8^{\frac{1}{3}}\left(1-\frac{3}{8}x^2\right)^{\frac{1}{3}} \\ &= 2\left\{1+\left(\frac{1}{3}\right)\left(-\frac{3}{8}x^2\right)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(-\frac{3}{8}x^2\right)^2+\ldots\right\} \\ &= 2-\frac{1}{4}x^2-\frac{1}{32}x^4+\ldots\end{aligned}$$

Valid for $\left|-\frac{3}{8}x^2\right| < 1$

and so for $|x^2| < \frac{8}{3}$

$|x| < \sqrt{\frac{8}{3}}$

c $$\begin{aligned}(3-x^2)^{\frac{5}{2}} &= \left[3\left(1-\frac{x^2}{3}\right)\right]^{\frac{5}{2}} \\ &= 3^{\frac{5}{2}}\left\{\left(1-\frac{x^2}{3}\right)^{\frac{5}{2}}\right\} \\ &= 9\sqrt{3}\left\{1+\left(\frac{5}{2}\right)\left(-\frac{x^2}{3}\right)+\frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)}{2!}\left(-\frac{x^2}{3}\right)^2+\ldots\right\} \\ &= 9\sqrt{3}-15\frac{\sqrt{3}}{2}x^2+15\frac{\sqrt{3}}{8}x^4+\ldots\end{aligned}$$

Valid for $\left|-\frac{x^2}{3}\right| < 1$

and so for $|x^2| < 3$

$|x| < \sqrt{3}$

3 $$\begin{aligned}(8-2x)^{\frac{1}{3}} &= \left[8\left(1-\frac{2}{3}x\right)\right]^{\frac{1}{3}} \\ &= 8^{\frac{1}{3}}\left(1-\frac{2}{3}x\right)^{\frac{1}{3}} \\ &= 2\left\{1+\left(\frac{1}{3}\right)\left(-\frac{2}{3}x\right)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(-\frac{2}{3}x\right)^2\right. \\ &\quad \left.+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(-\frac{2}{3}x\right)^3+\ldots\right\} \\ &= 2-\frac{1}{6}x-\frac{1}{72}x^2-\frac{5}{2592}x^3-\ldots\end{aligned}$$

$$\begin{aligned}&(1-2x)\left(2-\frac{1}{6}x-\frac{1}{72}x^2-\frac{5}{2592}x^3-\ldots\right) \\ &= 2+\left(-\frac{1}{6}-4\right)x+\left(-\frac{1}{72}+\frac{2}{6}\right)x^2+\left(-\frac{5}{2592}+\frac{2}{72}\right)x^3+\ldots \\ &= 2-\frac{25}{6}x+\frac{23}{72}x^2+\frac{67}{2592}x^3+\ldots\end{aligned}$$

4 **a** $(2-x)^{-1} = 2^{-1}\left(1-\frac{x}{2}\right)^{-1}$

$$= \frac{1}{2}\left\{1+(-1)\left(-\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2+...\right\}$$

$$= \frac{1}{2}+\frac{1}{4}x+\frac{1}{8}x^2+...$$

$$(1+3x)^{-2} = 1+(-2)(3x)+\frac{(-2)(-3)}{2!}(3x)^2+...$$

$$= 1-6x+27x^2-...$$

b Using the results from part **a**:

$$(2-x)^{-1}(1+3x)^{-2} = \left(\frac{1}{2}+\frac{1}{4}x+\frac{1}{8}x^2+...\right)(1-6x+27x^2-...)$$

$$= \frac{1}{2}+\left(\frac{1}{4}-3\right)x+\left(\frac{27}{2}-\frac{6}{4}+\frac{1}{8}\right)x^2+...$$

$$= \frac{1}{2}-\frac{11}{4}x+\frac{97}{8}x^2+...$$

Valid for $\left|\frac{x}{2}\right|<1$ and $|3x|<1$

so valid for $|x|<2$ and $|x|<\frac{1}{3}$

so valid for $|x|<\frac{1}{3}$.

5 $(a-5x)^{-2} = a^{-2}\left(1-\frac{5}{a}x\right)^{-2}$

$$= \frac{1}{a^2}\left\{1+(-2)\left(-\frac{5}{a}x\right)+\frac{(-2)(-3)}{2!}\left(-\frac{5}{a}x\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(-\frac{5}{a}x\right)^3...\right\}$$

$$= \frac{1}{a^2}+\frac{10}{a^3}x+\frac{75}{a^4}x^2+...$$

$$= \frac{1}{4}+\frac{5}{4}x+bx^2$$

Equating coefficients of x:

$$\frac{10}{a^3} = \frac{5}{4}$$

$$a^3 = 8$$

$$a = 2$$

Equating coefficients of x^2:

$$b = \frac{75}{a^4} = \frac{75}{16}$$

Term in x^3:

$$\frac{1}{a^2}\frac{(-2)(-3)(-4)}{3!}\left(-\frac{5}{a}x\right)^3 = \frac{4\times 125}{a^5}x^3 = \frac{500}{32}x^3$$

$$= \frac{125}{8}x^3$$

Note that at the point where $\frac{1}{a^2}+\frac{10}{a^3}x+\frac{75}{a^4}x^2+....= \frac{1}{4}+\frac{5}{4}x+bx^2$, you could look at solving $\frac{1}{a^2}=\frac{1}{4}$. This gives $a=\pm 2$. You need to look at the next term to confirm that only the positive value is correct.

6 $(3+ax)^{\frac{1}{2}} = 3^{\frac{1}{2}}\left(1+\frac{a}{3}x\right)^{\frac{1}{2}}$

$$= \sqrt{3}\left\{1+\left(\frac{1}{2}\right)\left(\frac{a}{3}x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{a}{3}x\right)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{a}{3}x\right)^3+\ldots\right\}$$

Using the fact that the coefficient of x^2 is three times the coefficient of x^3:

$$-\frac{a^2\sqrt{3}}{72} = 3\times\frac{\sqrt{3}}{16\times 27}a^3$$

$$a=-2$$

7 **a** $\left(1+\frac{2}{x}\right)^{-1} = 1+(-1)\left(\frac{2}{x}\right)+\frac{(-1)(-2)}{2!}\left(\frac{2}{x}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{2}{x}\right)^3+\ldots$

$$= 1-\frac{2}{x}+\frac{4}{x^2}-\frac{8}{x^3}+\ldots$$

b $\left(1+\frac{2}{x}\right)^{-1} = \left(\frac{x+2}{x}\right)^{-1}$

$$= \frac{x}{x+2}$$

and

$$\left(1+\frac{2}{x}\right)^{-1} = \left[\frac{2}{x}\left(\frac{x}{2}+1\right)\right]^{-1}$$

$$= \left(\frac{2}{x}\right)^{-1}\left(1+\frac{x}{2}\right)^{-1}$$

$$= \frac{x}{2}\left(1+\frac{x}{2}\right)^{-1}$$

c $\frac{x}{2}\left(1+\frac{x}{2}\right)^{-1} = \frac{x}{2}\left\{1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right\}$

$$= \frac{x}{2}-\frac{x^2}{4}+\frac{x^3}{8}-\frac{x^4}{16}+\ldots$$

d They are valid for different ranges of x, so the expansion is different.

EXERCISE 7E

1 **a** $\dfrac{7x-1}{(1-x)(1+2x)} = \dfrac{A}{1-x} + \dfrac{B}{1+2x}$

$$7x - 1 = A(1+2x) + B(1-x)$$

Equating coefficients of x:

$7 = 2A - B$ .... [1]

Equating constant terms:

$-1 = A + B$ ...... [2]

[2] + [1]:

$6 = 3A$

$A = 2$

Substituting into [1]:

$7 = 4 - B$

$B = -3$

$$\frac{7x-1}{(1-x)(1+2x)} = \frac{2}{1-x} - \frac{3}{1+2x}$$

b

Expand the two terms separately and then combine the results. If you try to do it all in one go, you will end up with far too much crammed onto one line.

$$\begin{aligned}\frac{2}{1-x} &= 2(1-x)^{-1}\\ &= 2\left\{1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \frac{(-1)(-2)(-3)}{3!}(-x)^3 + \ldots\right\}\\ &= 2 + 2x + 2x^2 + 2x^3 + \ldots\\ \frac{3}{1+2x} &= 3(1+2x)^{-1}\\ &= 3\left\{1 + (-1)(2x) + \frac{(-1)(-2)}{2!}(2x)^2 + \frac{(-1)(-2)(-3)}{3!}(2x)^3 + \ldots\right\}\\ &= 3 - 6x + 12x^2 - 24x^3 + \ldots\\ \frac{2}{1-x} - \frac{3}{1+2x} &= 2 + 2x + 2x^2 + 2x^3 + \ldots - \left(3 - 6x + 12x^2 - 24x^3 + \ldots\right)\\ &= -1 + 8x - 10x^2 + 26x^3 - \ldots\end{aligned}$$

2 **a** $\dfrac{5x^2+x}{(1-x)^2(1-3x)} = \dfrac{A}{1-x} + \dfrac{B}{(1-x)^2} + \dfrac{C}{1-3x}$

$$5x^2 + x = A(1-x)(1-3x) + B(1-3x) + C(1-x)^2$$

Letting $x = 1$:

$6 = 0 - 2B + 0$

$B = -3$

Letting $x = \dfrac{1}{3}$:

$\dfrac{8}{9} = 0 + 0 + \dfrac{4}{9}C$

$C = 2$

Equating coefficients of x^2:

$5 = 3A + C$

$5 = 3A + 2$

$3A = 3$

$A = 1$

$$\frac{5x^2+x}{(1-x)^2(1-3x)} = \frac{1}{1-x} - \frac{3}{(1-x)^2} + \frac{2}{1-3x}$$

b $\dfrac{1}{1-x} = (1-x)^{-1}$

$= 1 + (-1)(-x) + \dfrac{(-1)(-2)}{2!}(-x)^2 + \dfrac{(-1)(-2)(-3)}{3!}(-x)^3 + ...$

$= 1 + x + x^2 + x^3 + ...$

$\dfrac{3}{(1-x)^2} = 3(1-x)^{-2}$

$= 3\left\{1 + (-2)(-x) + \dfrac{(-2)(-3)}{2!}(-x)^2 + \dfrac{(-2)(-3)(-4)}{3!}(-x)^3 + ...\right\}$

$= 3 + 6x + 9x^2 + 12x^3 + ...$

$\dfrac{2}{1-3x} = 2(1-3x)^{-1}$

$= 2\left\{1 + (-1)(-3x) + \dfrac{(-1)(-2)}{2!}(-3x)^2 + \dfrac{(-1)(-2)(-3)}{3!}(-3x)^3 + ...\right\}$

$= 2 + 6x + 18x^2 + 54x^3 + ...$

So $\dfrac{1}{1-x} - \dfrac{3}{(1-x)^2} + \dfrac{2}{1-3x}$

$= 1 + x + x^2 + x^3 + ... - \left(3 + 6x + 9x^2 + 12x^3 + ...\right) + \left(2 + 6x + 18x^2 + 54x^3 + ...\right)$

$= x + 10x^2 + 43x^3 + ...$

3 **a** $\dfrac{7x^2 + 4x + 4}{(1-x)(2x^2+1)} = \dfrac{A}{1-x} + \dfrac{Bx + C}{2x^2+1}$

$7x^2 + 4x + 4 = A(2x^2+1) + (Bx + C)(1-x)$

Letting $x = 1$:

$15 = 3A + 0$

$A = 5$

Letting $x = 0$:

$4 = A + C = 5 + C$

$C = -1$

Equating coefficients of x^2:

$7 = 2A - B = 10 - B$

$B = 3$

$\dfrac{7x^2 + 4x + 4}{(1-x)(2x^2+1)} = \dfrac{5}{1-x} + \dfrac{3x-1}{2x^2+1}$

b $\dfrac{5}{1-x} + \dfrac{3x-1}{2x^2+1} = 5(1-x)^{-1} + (3x-1)(2x^2+1)^{-1}$

$(1-x)^{-1} = 1 + (-1)(-x) + \dfrac{(-1)(-2)}{2!}(-x)^2 + \dfrac{(-1)(-2)(-3)}{3!}(-x)^3 + ...$

$= 1 + x + x^2 + x^3 + ...$

$(2x^2+1)^{-1} = (1+2x^2)^{-1}$

$= 1 + (-1)(2x^2) + \dfrac{(-1)(-2)}{2!}(2x^2)^2 + ...$

$= 1 - 2x^2 + 4x^4 + ...$

$5(1-x)^{-1} + (3x-1)(2x^2+1)^{-1}$

$= 5\left\{1 + x + x^2 + x^3 + ...\right\} + (3x-1)\left\{1 - 2x^2 + 4x^4 + ...\right\}$

$= 5 + 5x + 5x^2 + 5x^3 + ... + 3x - 1 - 6x^3 + 2x^2 + 12x^5 - 4x^4 + ...$

$= 4 + 8x + 7x^2 - x^3 + ...$

4 **a** Dividing $-6x^2 - 7x + 19$ by $-6x^2 + x + 2$ using long division:

$(2x+1)(2-3x) = -6x^2 + x + 2$

$$\begin{array}{r}
1 \\
-6x^2 + x + 2 \overline{)\,-6x^2 - 7x + 19} \\
\underline{-6x^2 + x + 2} \\
-8x + 17
\end{array}$$

$$\frac{19-7x-6x^2}{(2x+1)(2-3x)}=1+\frac{17-8x}{(2x+1)(2-3x)}$$

Splitting the proper fraction $\frac{17-8x}{(2x+1)(2-3x)}$ into partial fractions:

$$\frac{17-8x}{(2x+1)(2-3x)}=\frac{A}{2x+1}+\frac{B}{2-3x}$$

$$17-8x=A(2-3x)+B(2x+1)$$

Letting $x=-\frac{1}{2}$:

$$21=\frac{7}{2}A+0$$

$$A=6$$

Equating constant terms:

$$17=2A+B$$

$$17=12+B$$

$$B=5$$

So $\frac{19-7x-6x^2}{(2x+1)(2-3x)}=1+\frac{6}{2x+1}+\frac{5}{2-3x}$

Although you could have substituted $x=\frac{2}{3}$ to get B, this would have involved more fractions calculations. Once you have A it is simpler to compare coefficients to get B in this case.

b $1+\frac{6}{2x+1}+\frac{5}{2-3x}=1+6(2x+1)^{-1}+5(2-3x)^{-1}$

$$=1+6(2x+1)^{-1}+5(2^{-1})\left(1-\frac{3}{2}x\right)^{-1}$$

$$=1+6(2x+1)^{-1}+\frac{5}{2}\left(1-\frac{3}{2}x\right)^{-1}$$

Coefficient of $x^2=6\left(\frac{(-1)(-2)}{2!}(2^2)\right)+\frac{5}{2}\left[\frac{(-1)(-2)}{2!}\left(-\frac{3}{2}\right)^2\right]$

$$=24+\frac{45}{8}=\frac{237}{8}$$

5 **a** $\frac{21}{(x-4)(x+3)}=\frac{A}{x-4}+\frac{B}{x+3}$

$$21=A(x+3)+B(x-4)$$

Letting $x=-3$:

$$21=0-7B$$

$$B=-3$$

Letting $x=4$:

$$21=7A$$

$$A=3$$

$$\frac{21}{(x-4)(x+3)}=\frac{3}{x-4}-\frac{3}{x+3}$$

b $\frac{3}{x-4}-\frac{3}{x+3}=3(x-4)^{-1}-3(x+3)^{-1}$

$$(x-4)^{-1}=-4^{-1}\left(1-\frac{x}{4}\right)^{-1}$$

$$=-\frac{1}{4}\left\{1+(-1)\left(-\frac{x}{4}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{x}{4}\right)^2+\ldots\right\}$$

$$=-\frac{1}{4}-\frac{x}{16}-\frac{x^2}{64}+\ldots$$

$$(x+3)^{-1} = 3^{-1}\left(1+\frac{x}{3}\right)^{-1}$$
$$= \frac{1}{3}\left\{1+(-1)\left(\frac{x}{3}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{3}\right)^2+...\right\}$$
$$= \frac{1}{3}-\frac{x}{9}+\frac{x^2}{27}+...$$

So $3(x-4)^{-1}-3(x+3)^{-1}$

$$= 3\left(-\frac{1}{4}-\frac{x}{16}-\frac{x^2}{64}+...\right)-3\left(\frac{1}{3}-\frac{x}{9}+\frac{x^2}{27}+...\right)$$
$$= -\frac{3}{4}-1+\left(-\frac{3}{16}+\frac{3}{9}\right)x+\left(-\frac{3}{64}-\frac{3}{27}\right)x^2+...$$
$$= -\frac{7}{4}+\frac{7}{48}x-\frac{91}{576}x^2+...$$

6 **a** $\frac{6x^2-24x+15}{(x+1)(x-2)^2} = \frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$

$$6x^2-24x+15 = A(x-2)^2+B(x+1)(x-2)+C(x+1)$$

Letting $x=2$:

$$-9 = 0+0+3C$$
$$C=-3$$

Letting $x=-1$:

$$45 = 9A+0+0$$
$$A=5$$

Equating coefficients of x^2:

$$6 = A+B = 5+B$$
$$B=1$$

$$\frac{6x^2-24x+15}{(x+1)(x-2)^2} = \frac{5}{x+1}+\frac{1}{x-2}-\frac{3}{(x-2)^2}$$

b $\frac{5}{x+1}+\frac{1}{x-2}-\frac{3}{(x-2)^2} = 5(1+x)^{-1}+(x-2)^{-1}-3(x-2)^{-2}$

$$= 5(1+x)^{-1}-\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}-\frac{3}{4}\left(1-\frac{x}{2}\right)^{-2}$$

$$(1+x)^{-1} = 1+(-1)x+\frac{(-1)(-2)}{2!}x^2+...$$
$$= 1-x+x^2-...$$

$$\left(1-\frac{x}{2}\right)^{-1} = 1+(-1)\left(-\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2+...$$
$$= 1+\frac{x}{2}+\frac{x^2}{4}+...$$

$$\left(1-\frac{x}{2}\right)^{-2} = 1+(-2)\left(-\frac{x}{2}\right)+\frac{(-2)(-3)}{2!}\left(-\frac{x}{2}\right)^2+...$$
$$= 1+x+\frac{3}{4}x^2+...$$

So $5(1+x)^{-1}-\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}-\frac{3}{4}\left(1-\frac{x}{2}\right)^{-2}$

$$= 5\left(1-x+x^2-...\right)-\frac{1}{2}\left(1+\frac{x}{2}+\frac{x^2}{4}+...\right)-\frac{3}{4}\left(1+x+\frac{3}{4}x^2+...\right)$$
$$= \frac{15}{4}-6x+\frac{69}{16}x^2-...$$

END-OF-CHAPTER REVIEW EXERCISE 7

P3 **This exercise is for Pure Mathematics 3 students only.**

1 $(1-2x)^{-4} = 1+(-4)(-2x)+\frac{(-4)(-5)}{2!}(-2x)^2+\frac{(-4)(-5)(-6)}{3!}(-2x)^3+...$
$= 1+8x+40x^2+160x^3+...$

2 $(1-6x)^{\frac{1}{3}} = 1+\left(\frac{1}{3}\right)(-6x)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(-6x)^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(-6x)^3+...$
$= 1-2x-4x^2-\frac{40}{3}x^3+...$

3 $(1+2x)^{-\frac{3}{2}} = 1+\left(-\frac{3}{2}\right)(2x)+\frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{2!}(2x)^2+...$
$= 1-3x+\frac{15}{2}x^2+...$

$(2-x)(1+2x)^{-\frac{3}{2}} = (2-x)\left(1-3x+\frac{15}{2}x^2+...\right)$
$= 2-x-6x+3x^2+15x^2-\frac{15}{2}x^3+...$
$= 2-7x+18x^2-...$

4 $(1+2x)^{-\frac{1}{2}} = 1+\left(-\frac{1}{2}\right)(2x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(2x)^2+...$
$= 1-x+\frac{3}{2}x^2-...$

$(1+3x)(1+2x)^{-\frac{1}{2}} = (1+3x)\left(1-x+\frac{3}{2}x^2-...\right)$
$= 1+3x-x-3x^2+...$
$= 1+2x-3x^2+\frac{3}{2}x^2-...$
$= 1+2x-\frac{3}{2}x^2+...$

5 $(1+3x)^{-\frac{1}{2}} = 1+\left(-\frac{1}{2}\right)(3x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(3x)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(3x)^3+...$
$= 1-\frac{3}{2}x+\frac{27}{8}x^2-\frac{135}{16}x^3+...$

6 $10(2-x)^{-2} = 10\left(2^{-2}\right)\left(1-\frac{x}{2}\right)^{-2}$
$= \frac{5}{2}\left\{1+(-2)\left(-\frac{x}{2}\right)+\frac{(-2)(-3)}{2!}\left(-\frac{x}{2}\right)^2+...\right\}$
$= \frac{5}{2}\left(1+x-\frac{3}{4}x^2+...\right)$
$= \frac{5}{2}+\frac{5}{2}x+\frac{15}{8}x^2+....$

7 $(4-5x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1-\frac{5}{4}x\right)^{-\frac{1}{2}}$
$= \frac{1}{2}\left\{1+\left(-\frac{1}{2}\right)\left(-\frac{5}{4}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{5}{4}x\right)^2+...\right\}$
$= \frac{1}{2}+\frac{5}{16}x+\frac{75}{256}x^2+...$

8 $$\frac{12}{x^2(2x-3)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{2x-3}$$
$$12=Ax(2x-3)+B(2x-3)+Cx^2$$
Letting $x=0$:

$12=0-3B+0$
$B=-4$

Letting $x=\frac{3}{2}$:

$12=0+0+\frac{9}{4}C$

$C=\frac{16}{3}$

Equating coefficients of x^2:

$0=2A+C$

$2A=-\frac{16}{3}$

$A=-\frac{8}{3}$

$$\frac{12}{x^2(2x-3)}=-\frac{8}{3x}-\frac{4}{x^2}+\frac{16}{3(2x-3)}$$

9 $$\frac{8x^2+4x+21}{(x+2)(x^2+5)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+5}$$
$$8x^2+4x+21=A\left(x^2+5\right)+(Bx+C)(x+2)$$
Letting $x=-2$:

$32-8+21=9A+0$
$9A=45$
$A=5$

Letting $x=0$:

$21=5A+2C$
$2C=21-25$
$C=-2$

Equating coefficients of x^2:

$8=A+B=5+B$
$B=3$

$$\frac{8x^2+4x+21}{(x+2)(x^2+5)}=\frac{5}{x+2}+\frac{3x-2}{x^2+5}$$

10 $$\frac{7x^2-3x+2}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$
$$7x^2-3x+2=A\left(x^2+1\right)+(Bx+C)x$$
Letting $x=0$:

$2=A$

Equating coefficients of x^2:

$7=A+B=2+B$
$B=5$

Equating coefficients of x:

$-3=C$

$$\frac{7x^2-3x+2}{x(x^2+1)}=\frac{2}{x}+\frac{5x-3}{x^2+1}$$

11 When the fraction is improper you usually use algebraic division before working out partial fractions. In Question **11**, the partial fraction form is given, so you can just multiply through by the denominators and use a combination of substitution and equating coefficients.

$$\frac{9x^3 - 11x^2 + 8x - 4}{x^2(3x-2)} = A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{3x-2}$$

$$9x^3 - 11x^2 + 8x - 4 = Ax^2(3x-2) + Bx(3x-2) + C(3x-2) + Dx^2$$

Letting $x = 0$:

$$-4 = 0 + 0 - 2C + 0$$
$$C = 2$$

Letting $x = \frac{2}{3}$:

$$\frac{8}{3} - \frac{44}{9} + \frac{16}{3} - 4 = 0 + 0 + 0 + \frac{4}{9}D$$
$$-\frac{8}{9} = \frac{4}{9}D$$
$$D = -2$$

Equating coefficients of x^3:

$$9 = 3A$$
$$A = 3$$

Equating coefficients of x^2:

$$-11 = -2A + 3B + D$$
$$-11 = -6 + 3B - 2$$
$$3B = -3$$
$$B = -1$$

$$\frac{9x^3 - 11x^2 + 8x - 4}{x^2(3x-2)} = 3 - \frac{1}{x} + \frac{2}{x^2} - \frac{2}{3x-2}$$

12 $(x+2)(2x-1) = 2x^2 + 3x - 2$

The fraction is improper, so dividing first:

$$\begin{array}{r} 2 \\ 2x^2+3x-2\overline{\big)4x^2-5x+3} \\ \underline{4x^2+6x-4} \\ -11x+7 \end{array}$$

$$\frac{4x^2 - 5x + 3}{(x+2)(2x-1)} = 2 + \frac{7 - 11x}{(x+2)(2x-1)}$$

Splitting the proper fraction $\frac{7-11x}{(x+2)(2x-1)}$ into partial fractions:

$$\frac{7-11x}{(x+2)(2x-1)} = \frac{A}{x+2} + \frac{B}{2x-1}$$

$$7 - 11x = A(2x-1) + B(x+2)$$

Letting $x = -2$:

$$29 = -5A$$
$$A = -\frac{29}{5}$$

Letting $x = \frac{1}{2}$:

$$\frac{3}{2} = 0 + \frac{5}{2}B$$
$$B = \frac{3}{5}$$

So $\frac{4x^2 - 5x + 3}{(x+2)(2x-1)} = 2 - \frac{29}{5(x+2)} + \frac{3}{5(2x-1)}$

13 $\left(1 - 2x^2\right)^{-2}$

$$= 1 + (-2)\left(-2x^2\right) + \frac{(-2)(-3)}{2!}\left(-2x^2\right)^2 + \ldots$$
$$= 1 + 4x^2 + 12x^4 + \ldots$$

$$\left(1+6x^2\right)^{\frac{2}{3}} = 1+\left(\frac{2}{3}\right)\left(6x^2\right)+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{2!}\left(6x^2\right)^2+...$$
$$= 1+4x^2-8x^4+...$$
$$\left(1-2x^2\right)^{-2}-\left(1+6x^2\right)^{\frac{2}{3}} = \left(1+4x^2+12x^4+...\right)-\left(1+4x^2-4x^4+...\right)$$
$$= 16x^4+...$$

So $k=6$

14 **i** $\mathrm{f}(x)=\frac{4x^2+12}{(x+1)(x-3)^2}=\frac{A}{x+1}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$

$4x^2+12=A(x-3)^2+B(x+1)(x-3)+C(x+1)$

Letting $x=3$:

$48=0+0+4C$
$C=12$

Letting $x=-1$:

$16=16A+0+0$
$A=1$

Equating coefficients of x^2:

$4=A+B=1+B$
$B=3$

$$\mathrm{f}(x)=\frac{4x^2+12}{(x+1)(x-3)^2}=\frac{1}{x+1}+\frac{3}{x-3}+\frac{12}{(x-3)^2}$$

ii $(1+x)^{-1}=(1+x)^{-1}=1+(-1)x+\frac{(-1)(-2)}{2!}x^2+...$

$$=1-x+x^2-...$$

$$3(x-3)^{-1}=3\left(-3^{-1}\right)\left(1-\frac{x}{3}\right)^{-1}$$
$$=-\left\{1+(-1)\left(-\frac{x}{3}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{x}{3}\right)^2+...\right\}$$
$$=-1-\frac{x}{3}-\frac{x^2}{9}+...$$

$$12(x-3)^{-2}=12(-3)^{-2}\left(1-\frac{x}{3}\right)^{-2}$$
$$=\frac{4}{3}\left\{1+(-2)\left(-\frac{x}{3}\right)+\frac{(-2)(-3)}{2!}\left(-\frac{x}{3}\right)^2+...\right\}$$
$$=\frac{4}{3}+\frac{8}{9}x+\frac{4}{9}x^2+...$$

So $\mathrm{f}(x)=\frac{1}{x+1}+\frac{3}{x-3}+\frac{12}{(x-3)^2}$

$$=\left(1-x+x^2-...\right)+\left(-1-\frac{x}{3}-\frac{x^2}{9}+...\right)+\left(\frac{4}{3}+\frac{8}{9}x+\frac{4}{9}x^2+...\right)$$
$$=\frac{4}{3}-\frac{4}{9}x+\frac{4}{3}x^2-...$$

15 **i** $\frac{2x^2-7x-1}{(x-2)\left(x^2+3\right)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+3}$

$2x^2-7x-1=A\left(x^2+3\right)+(Bx+C)(x-2)$

Letting $x=2$:

$-7=7A+0$
$A=-1$

Letting $x=0$:

$-1=3A-2C$
$-1=-3-2C$
$2C=-2$
$C=-1$

Equating coefficients of x^2:

$2 = A + B = -1 + B$

$B = 3$

$$\frac{2x^2 - 7x - 1}{(x-2)(x^2+3)} = -\frac{1}{x-2} + \frac{3x-1}{x^2+3}$$

ii $-\frac{1}{x-2} + \frac{3x-1}{x^2+3}$

$$(x-2)^{-1} = (-2)^{-1}\left(1 - \frac{x}{2}\right)^{-1}$$

$$= -\frac{1}{2}\left\{1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2 + \dots\right\}$$

$$= -\frac{1}{2} - \frac{x}{4} - \frac{1}{8}x^2 + \dots$$

$$\left(x^2+3\right)^{-1} = 3^{-1}\left(1 + \frac{x^2}{3}\right)^{-1}$$

$$= \frac{1}{3}\left\{1 + (-1)\left(\frac{x^2}{3}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x^2}{3}\right)^2 + \dots\right\}$$

$$= \frac{1}{3} - \frac{x^2}{9} + \frac{x^4}{27} - \dots$$

$$(3x-1)\left(x^2+3\right)^{-1} = (3x-1)\left(\frac{1}{3} - \frac{x^2}{9} + \frac{x^4}{27} - \dots\right)$$

$$= x - \frac{1}{3} - \frac{x^3}{3} + \frac{x^2}{9} + \dots$$

$$= -\frac{1}{3} + x + \frac{x^2}{9} - \dots$$

So $\mathrm{f}(x) = -\frac{1}{x-2} + \frac{3x-1}{x^2+3}$

$$= -\left(-\frac{1}{2} - \frac{x}{4} - \frac{1}{8}x^2 + \dots\right) + \left(-\frac{1}{3} + x + \frac{x^2}{9} - \dots\right)$$

$$= \frac{1}{6} + \frac{5}{4}x + \frac{17}{72}x^2 + \dots$$

16 **i** $\frac{3x}{(1+x)(1+2x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+2x^2}$

$$3x = A\left(1+2x^2\right) + (Bx+C)(1+x)$$

Letting $x = -1$:

$-3 = 3A + 0$

$A = -1$

Letting $x = 0$:

$0 = A + C$

$C = 1$

Equating coefficients of x^2:

$0 = 2A + B$

$B = 2$

$$\frac{3x}{(1+x)(1+2x^2)} = -\frac{1}{1+x} + \frac{2x+1}{1+2x^2}$$

ii $(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \dots$

$$= 1 - x + x^2 - x^3 + \dots$$

$$\left(1+2x^2\right)^{-1} = 1 + (-1)\left(2x^2\right) + \frac{(-1)(-2)}{2!}\left(2x^2\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(2x^2\right)^3 + \dots$$

$$= 1 - 2x^2 + 4x^4 - 8x^6 + \dots$$

$$(2x+1)\left(1+2x^2\right)^{-1} = (2x+1)\left(1 - 2x^2 + 4x^4 - 8x^6 + \dots\right)$$

$$= 2x + 1 - 4x^3 - 2x^2 + \dots$$

$$= 1 + 2x - 2x^2 - 4x^3 + \dots$$

$$\text{So } \mathrm{f}(x) = -\frac{1}{1+x} + \frac{2x+1}{1+2x^2}$$
$$= -\left(1 - x + x^2 - x^3 + \ldots\right) + \left(1 + 2x - 2x^2 - 4x^3 + \ldots\right)$$
$$= 3x - 3x^2 - 3x^3 + \ldots$$

Chapter 8
Further calculus

P3 **This chapter is for Pure Mathematics 3 students only.**

EXERCISE 8A

1 **b** Using the chain rule:

$$\frac{d}{dx}(\tan^{-1} 5x)$$
$$= \frac{1}{1+(5x)^2} \times \frac{d}{dx}(5x)$$
$$= \frac{5}{1+25x^2}$$

f Using the chain rule and then the quotient rule:

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{2x}{x+1}\right)\right)$$
$$= \frac{1}{1+\left(\frac{2x}{x+1}\right)^2} \times \frac{d}{dx}\left(\frac{2x}{x+1}\right)$$
$$= \frac{1}{1+\left(\frac{2x}{x+1}\right)^2} \times \frac{(x+1)(2)-2x(1)}{(x+1)^2}$$
$$= \frac{2x+2-2x}{(x+1)^2+(2x)^2}$$
$$= \frac{2}{x^2+2x+1+4x^2}$$
$$= \frac{2}{5x^2+2x+1}$$

2 **a** Using the product rule:

$$\frac{d}{dx}(x\tan^{-1}x) = (1)\ \tan^{-1} x + x \times \frac{1}{1+x^2}$$
$$= \tan^{-1} x + \frac{x}{1+x^2}$$

b Using the quotient rule:

$$\frac{d}{dx}\left(\frac{\tan^{-1} 2x}{x}\right)$$
$$= \frac{x\left(\frac{1}{1+(2x)^2}\times 2\right)-(1)\tan^{-1} 2x}{x^2}$$
$$= \frac{\frac{2x}{1+4x^2}-\tan^{-1} 2x}{x^2}$$
$$= \frac{2}{x(1+4x^2)} - \frac{1}{x^2}\tan^{-1}\ 2x$$
$$= \frac{2x-(4x^2+1)\tan^{-1} 2x}{x^2(1+4x^2)}$$

c Using the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x \tan^{-1} x)$$
$$= \mathrm{e}^x \times \frac{1}{1+x^2} + \mathrm{e}^x \tan^{-1} x$$
$$= \mathrm{e}^x \left(\frac{1}{1+x^2} + \tan^{-1} x\right)$$

3 Using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+\left(\frac{x}{2}\right)^2} \times \frac{1}{2} = \frac{1}{2+\frac{x^2}{2}}$$
$$= \frac{2}{4+x^2}$$

When $x = 2$:

$$y = \tan^{-1} 1 = \frac{\pi}{4}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{4+4} = \frac{1}{4}$$

Equation of tangent is:

$$y - \frac{\pi}{4} = \frac{1}{4}(x-2)$$
$$4y - \pi = x - 2$$
$$x - 4y = 2 - \pi$$

4
$$y = \tan^{-1} x$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$$

when $x = -1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+1} = \frac{1}{2}$$

When $x = 1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+1} = \frac{1}{2}$$

Gradient of the normal at this point is $-\dfrac{1}{\left(\frac{1}{2}\right)} = -2$

(gradient of tangent at $x = -1$) $\times$ (gradient of normal at $x = 1$)

$$= \frac{1}{2} \times -2 = -1$$

The tangent at $x = -1$ is perpendicular to the normal at $x = 1$.

$x = -1$ gives $y = \tan^{-1}(-1) = -\dfrac{\pi}{4}$

Equation of tangent is:

$$y - \left(-\frac{\pi}{4}\right) = \frac{1}{2}(x+1)$$
$$y + \frac{\pi}{4} = \frac{1}{2}x + \frac{1}{2}$$
$$4y + \pi = 2x + 2$$
$$2x - 4y = \pi - 2 \text{ .... [1]}$$

$x = 1$ gives $y = \tan^{-1}(1) = \dfrac{\pi}{4}$

Equation of normal is:

$$y - \frac{\pi}{4} = -2(x-1)$$
$$4y - \pi = -8x + 8$$
$$8x + 4y = \pi + 8 \text{ .... [2]}$$

[1] + [2]:

$$10x = 2\pi + 6$$
$$x = \frac{3+\pi}{5}$$

EXERCISE 8B

1 b $\int \frac{1}{16+x^2}\,dx = \int \frac{1}{x^2+4^2}\,dx$

$= \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + c$

e $\int \frac{1}{4x^2+3}\,dx = \frac{1}{4}\int \frac{1}{x^2+\frac{3}{4}}\,dx$

$= \frac{1}{4}\int \frac{1}{x^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$

$= \frac{1}{4}\left(\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right) + c$

$= \frac{\sqrt{3}}{6}\tan^{-1}\left(\frac{2\sqrt{3}x}{3}\right) + c$

2 a $\int_0^3 \frac{1}{x^2+9}\,dx = \int_0^3 \frac{1}{x^2+3^2}\,dx$

$= \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$

$= \frac{1}{3}\tan^{-1} 1 - \frac{1}{3}\tan^{-1} 0$

$= \frac{1}{3}\left(\frac{\pi}{4}\right) - 0$

$= \frac{\pi}{12}$

b $\int_0^{\frac{1}{2}} \frac{2}{4x^2+1}\,dx$

$= \frac{1}{4}\int_0^{\frac{1}{2}} \frac{2}{x^2+\frac{1}{4}}\,dx$

$= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{x^2+\frac{1}{4}}\,dx$

$= \frac{1}{2}\left[2\tan^{-1} 2x\right]_0^{\frac{1}{2}}$

$= \frac{1}{2}\left(2\tan^{-1} 1 - 2\tan^{-1} 0\right)$

$= \frac{\pi}{4}$

c $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{3x^2+2}\,dx$

$= \frac{1}{3}\int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{x^2+\frac{2}{3}}\,dx$

$= \frac{1}{3}\left[\sqrt{\frac{3}{2}}\tan^{-1}\left(\sqrt{\frac{3}{2}}x\right)\right]_{-\sqrt{2}}^{\sqrt{2}}$

$= \frac{1}{3}\left(\sqrt{\frac{3}{2}}\right)\left(\tan^{-1}\sqrt{3} - \tan^{-1}(-\sqrt{3})\right)$

$= \frac{1}{\sqrt{6}}\left(\frac{\pi}{3}+\frac{\pi}{3}\right)$

$= \frac{\sqrt{6}}{6}\left(\frac{2\pi}{3}\right)$

$= \frac{\sqrt{6}}{9}\pi$

3

Remember that if you rotate the region bounded by the x-axis, the curve with equation $y = \mathrm{f}(x)$ and the lines $x = a$ and $x = b$ then the volume generated is $V = \int_a^b \pi y^2 \, \mathrm{d}x$.

a

$$\begin{aligned} V &= \int_{-1}^{1} \pi y^2 \, \mathrm{d}x \\ &= \pi \int_{-1}^{1} \frac{4}{x^2+1} \, \mathrm{d}x \\ &= 4\pi \left[\tan^{-1} x\right]_{-1}^{1} \\ &= 4\pi \left(\tan^{-1} 1 - \tan^{-1}(-1)\right) \\ &= 4\pi \left(\frac{\pi}{4} + \frac{\pi}{4}\right) \\ &= \frac{4\pi^2}{2} = 2\pi^2 \end{aligned}$$

EXERCISE 8C

1 Remember that if the numerator is the derivative of the denominator then you can use logarithms. You can take out factors to make sure that the numerator matches the derivative of the denominator.

If $f(x) = x^3 - 1$ then $f'(x) = 3x^2$

$$\int \frac{6x^2}{x^3 - 1}\,dx$$
$$= 2\int \frac{3x^2}{x^3 - 1}\,dx$$
$$= 2\ln|x^3 - 1| + c$$

e If $f(x) = 2 - x^2$ then $f'(x) = -2x$

$$\int \frac{x}{2 - x^2}\,dx$$
$$= -\frac{1}{2}\int \frac{-2x}{2 - x^2}\,dx$$
$$= -\frac{1}{2}\ln|2 - x^2| + c$$

2 **b** If $f(x) = x^3 + 2$ then $f'(x) = 3x^2$

$$\int_0^2 \frac{3x^2}{x^3 + 2}\,dx$$
$$= \left[\ln|x^3 + 2|\right]_0^2$$
$$= \ln 10 - \ln 2$$
$$= \ln 5$$

e If $f(x) = \sin x$ then $f'(x) = \cos x$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos x}{\sin x}\,dx$$
$$= \left[\ln|\sin x|\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$
$$= \ln\left|\sin\left(\frac{\pi}{4}\right)\right| - \ln\left|\sin\left(\frac{\pi}{6}\right)\right|$$
$$= \ln\left(\frac{\sqrt{2}}{2}\right) - \ln\left(\frac{1}{2}\right)$$
$$= \ln\sqrt{2}$$
$$= \ln 2^{\frac{1}{2}}$$
$$= \frac{1}{2}\ln 2$$

3 If $f(x) = e^x - e^{-x}$ then $f'(x) = e^x + e^{-x}$

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$$
$$= \ln|e^x - e^{-x}| + c$$
$$= \ln\left(e^{-x}|e^{2x} - 1|\right) + c$$
$$= \ln e^{-x} + \ln|e^{2x} - 1| + c$$
$$= \ln|e^{2x} - 1| - x + c$$

If $x < 0$ then $e^{2x} - 1 < 0$ and $|e^{2x} - 1| = 1 - e^{2x}$

So $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \ln\left(1 - e^{2x}\right) - x + c$

Note that if $x > 0$ then $e^{2x} > 1 \Rightarrow 1 - e^{2x} < 0$ and the given answer requires the logarithm of a negative number.

4 $1 - 2\cos 2x$

$= 1 - 2(2\cos^2 x - 1)$

$= 3 - 4\cos^2 x$

So $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{\sin x \cos x}{1 - 2\cos 2x}\,dx$

$= -\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{\sin x \cos x}{3 - 4\cos^2 x}\,dx$

If $f(x) = 3 - 4\cos^2 x$ then $f'(x) = 8\cos x \sin x$

So $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{\sin x \cos x}{3 - 4\cos^2 x}\,dx$

$= \frac{1}{8}\left[\ln(3 - 4\cos^2 x)\right]_{\frac{\pi}{3}}^{\frac{\pi}{4}}$

$= \frac{1}{8}(\ln 1 - \ln 2)$

$= -\frac{1}{8}\ln 2$

5 If $f(x) = x^2 + 1$ then $f'(x) = 2x$

$\displaystyle\int_0^p \frac{4x}{x^2 + 1}\,dx = 4$

$2\displaystyle\int_0^p \frac{2x}{x^2 + 1}\,dx = 4$

$\left[\ln(x^2 + 1)\right]_0^p = 2$

$\ln(p^2 + 1) - \ln 1 = 2$

$p^2 + 1 = e^2$

$p^2 = e^2 - 1$

$p = \sqrt{e^2 - 1}$

EXERCISE 8D

1
$$\begin{aligned} u &= x^2 - 3 \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= 2x \\ \mathrm{d}u &= 2x\,\mathrm{d}x \\ x\,\mathrm{d}x &= \frac{1}{2}\,\mathrm{d}u \\ I &= \int \frac{x}{\sqrt{x^2-3}}\,\mathrm{d}x \\ &= \int \frac{1}{2\sqrt{u}}\,\mathrm{d}u \\ &= \int \frac{1}{2}u^{-\frac{1}{2}}\,\mathrm{d}u \\ &= u^{\frac{1}{2}} + c \\ &= \sqrt{x^2-3} + c \end{aligned}$$

2 **b**
$$\begin{aligned} u &= 1 - 2x^2 \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= -4x \\ \mathrm{d}u &= -4x\,\mathrm{d}x \\ x\,\mathrm{d}x &= -\frac{1}{4}\,\mathrm{d}u \end{aligned}$$
$$\begin{aligned} &\int x\sqrt{1-2x^2}\,\mathrm{d}x \\ &= \int -\frac{1}{4}\sqrt{u}\,\mathrm{d}u \\ &= -\frac{1}{4}\int u^{\frac{1}{2}}\,\mathrm{d}u \\ &= -\frac{1}{4}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c \\ &= -\frac{1}{6}(1-2x^2)^{\frac{3}{2}} + c \end{aligned}$$

e
$$\begin{aligned} u &= 5x + 1 \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= 5 \\ 5\,\mathrm{d}x &= \mathrm{d}u \\ x &= \frac{u-1}{5} \Rightarrow \frac{5x}{5x+1} = \frac{u-1}{u} \end{aligned}$$
$$\begin{aligned} &\int \frac{5x}{5x+1}\,\mathrm{d}x \\ &= \int \frac{u-1}{u} \times \frac{1}{5}\,\mathrm{d}u \\ &= \frac{1}{5}\int \left(1 - \frac{1}{u}\right)\mathrm{d}u \\ &= \frac{1}{5}(u - \ln\,|u|) + c \\ &= \frac{1}{5}(5x+1) - \frac{1}{5}\ln\,|5x+1| + c \\ &= x + \frac{1}{5} - \frac{1}{5}\ln\,|5x+1| + c \end{aligned}$$

3 Work through the differentiation part first and calculate new limits before you start to integrate. This means you can integrate without making any further calculations or rearrangements along the way.

$$x = \sin\theta$$
$$\frac{dx}{d\theta} = \cos\theta$$
$$dx = \cos\theta\, d\theta$$
$$x = 0 \Rightarrow \theta = 0$$
$$x = 1 \Rightarrow \theta = \frac{\pi}{2}$$
$$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\, dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\cos\theta} \cos\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$
$$= \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{4} - 0 - 0 + 0$$
$$= \frac{\pi}{4}$$

4 **b**

$$u = 3 - \sqrt{x} = 3 - x^{\frac{1}{2}}$$
$$\frac{du}{dx} = -\frac{1}{2}x^{-\frac{1}{2}} = -\frac{1}{2\sqrt{x}} = -\frac{1}{2(3-u)}$$
$$dx = -2(3-u)\, du$$
$$x = 0 \Rightarrow u = 3$$
$$x = 4 \Rightarrow u = 1$$
$$\int_0^4 \frac{\sqrt{x}}{3-\sqrt{x}}\, dx$$
$$= \int_3^1 \frac{(3-u)}{u} \times -2(3-u)\, du$$
$$= -2\int_3^1 \left(\frac{9-6u+u^2}{u}\right) du$$
$$= -2\int_3^1 \left(\frac{9}{u} - 6 + u\right) du$$
$$= -2\left[9\ \ln\ |u| - 6u + \frac{1}{2}u^2\right]_3^1$$
$$= -2\left\{9\ \ln\ 1 - 6 + \frac{1}{2} - \left(9\ \ln\ 3 - 18 + \frac{9}{2}\right)\right\}$$
$$= -2\left(0 - 6 + \frac{1}{2} - 9\ \ln\ 3 + 18 - \frac{9}{2}\right)$$
$$= -2(8 - 9\ \ln\ 3)$$
$$= 18\ \ln\ 3 - 16$$

g

$$x = 2\cos\theta$$
$$\frac{dx}{d\theta} = -2\sin\theta$$
$$dx = -2\sin\theta\, d\theta$$
$$\sqrt{4-x^2}$$
$$= \sqrt{4-4\cos^2\theta}$$
$$= \sqrt{4\sin^2\theta}$$
$$= 2\sin\theta$$

$x = 0 \Rightarrow \theta = \frac{\pi}{2}$

$x = 1 \Rightarrow \theta = \frac{\pi}{3}$

$$\int_0^1 \frac{x^2}{\sqrt{4 - x^2}}\,\mathrm{d}x$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \frac{4\cos^2\theta}{2\sin\theta}(-2\sin\theta)\,\mathrm{d}\theta$$

$$= -4\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \cos^2\theta\,\mathrm{d}\,\theta$$

$$= -4\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\,\mathrm{d}\theta$$

$$= -4\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{2}}^{\frac{\pi}{3}}$$

$$= -4\left(\frac{\pi}{6} + \frac{\sqrt{3}}{8} - \frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$$= \frac{1}{6}(2\pi - 3\sqrt{3})$$

When you see that the 'lower' limit is actually larger than the 'upper' limit, resist the temptation to swap them over. If you swap them, you will change the sign of the integral.

5 $u = x - 2$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$\mathrm{d}u = \mathrm{d}x$

$x = 1 \Rightarrow u = -1$

$x = 2 \Rightarrow u = 0$

$$\int_1^2 \frac{4}{1 + (x-2)^2}\,\mathrm{d}x$$

$$= \int_{-1}^0 \frac{4}{1 + u^2}\,\mathrm{d}u$$

$$= \left[4\tan^{-1} u\right]_{-1}^0$$

$$= 0 + \pi = \pi$$

6 $u = \sin x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$

$\cos x\,\mathrm{d}x = \mathrm{d}u$

$u^2 = \sin^2 x = 1 - \cos^2 x$

$\cos^2 x = 1 - u^2$

$\cos^3 x\,\mathrm{d}x = (1 - u^2)\,\mathrm{d}u$

$x = \frac{\pi}{2} \Rightarrow u = 1$

$x = 0 \Rightarrow u = 0$

$$\int_0^{\frac{\pi}{2}} 2\sin^2 x\cos^3 x\,\mathrm{d}x$$
$$= \int_0^1 2u^2(1-u^2)\,\mathrm{d}u$$
$$= \int_0^1 \left(2u^2 - 2u^4\right)\mathrm{d}u$$
$$= \left[\frac{2}{3}u^3 - \frac{2}{5}u^5\right]_0^1$$
$$= \frac{2}{3} - \frac{2}{5} - 0 + 0$$
$$= \frac{4}{15}$$

7

> Remember that if you rotate the region bounded by the x-axis, the curve with equation $y = \mathrm{f}(x)$ and the lines $x = a$ and $x = b$ then the volume generated is $V = \int_a^b \pi y^2\,\mathrm{d}x$.

$$u = \ln x$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$
$$\mathrm{d}u = \frac{1}{x}\mathrm{d}x$$
$$x = 1 \Rightarrow u = 0$$
$$x = \mathrm{e} \Rightarrow u = 1$$
$$\int_1^{\mathrm{e}} \pi\frac{(\ln x)^2}{x}\,\mathrm{d}x$$
$$= \int_0^1 \pi u^2\,\mathrm{d}u$$
$$= \pi\left[\frac{1}{3}u^3\right]_0^1$$
$$= \frac{1}{3}\pi$$

8

$$u = \cos x$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$$
$$\mathrm{d}u = -\sin x\,\mathrm{d}x$$
$$x = 0 \Rightarrow u = 1$$
$$x = \pi \Rightarrow u = -1$$

Using the fact that the curve has rotational symmetry of order 2 about the origin:

Areas enclosed $= 2 \times$ area from 0 to π

$$= 2\int_0^{\pi} 3\mathrm{e}^{\cos x}\sin x\,\mathrm{d}x$$
$$= -2\int_1^{-1} 3\mathrm{e}^u\,\mathrm{d}u$$
$$= -2[3\mathrm{e}^u]_1^{-1}$$
$$= -2\left(3\mathrm{e}^{-1} - 3\mathrm{e}\right)$$
$$= 6\mathrm{e} - \frac{6}{\mathrm{e}}$$

EXERCISE 8E

1 **b** $$\frac{2x-5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$2x-5 = A(x-1) + B(x+2)$$

Letting $x = 1$:

$-3 = 0 + 3B$

$B = -1$

Letting $x = -2$:

$-9 = -3A + 0$

$A = 3$

$$\frac{2x-5}{(x+2)(x-1)} = \frac{3}{x+2} - \frac{1}{x-1}$$

$$\int \frac{2x-5}{(x+2)(x-1)}\,\mathrm{d}x = \int \frac{3}{x+2} - \frac{1}{x-1}\,\mathrm{d}x$$

$$= 3\ln|x+2| - \ln|x-1| + c$$

d $$\frac{x^2+2x-5}{(x-3)(x^2+1)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1}$$

$$x^2+2x-5 = A(x^2+1) + (Bx+C)(x-3)$$

Letting $x = 3$:

$10 = 10A + 0$

$A = 1$

Letting $x = 0$:

$-5 = A - 3C = 1 - 3C$

$C = 2$

Equating coefficients of x^2:

$1 = A + B$

$B = 0$

$$\frac{x^2+2x-5}{(x-3)(x^2+1)} = \frac{1}{x-3} + \frac{2}{x^2+1}$$

$$\int \frac{x^2+2x-5}{(x-3)(x^2+1)}\,\mathrm{d}x = \int \left(\frac{1}{x-3} + \frac{2}{x^2+1}\right)\mathrm{d}x$$

$$= \ln|x-3| + 2\tan^{-1}x + c$$

When you need to integrate a fraction, remember to check if the numerator is the derivative of the denominator or some other standard form before rewriting with indices. In part **d**, the inverse tan integral appeared alongside a logarithm. This is common with partial fractions of this type.

2 **b** $$\frac{4x+5}{(2x+1)(x+2)} = \frac{A}{2x+1} + \frac{B}{x+2}$$

$$4x+5 = A(x+2) + B(2x+1)$$

Letting $x = -2$:

$-3 = 0 - 3B$

$B = 1$

Letting $x = -\frac{1}{2}$:

$3 = \frac{3}{2}A + 0$

$A = 2$

$$\frac{4x+5}{(2x+1)(x+2)} = \frac{2}{2x+1} + \frac{1}{x+2}$$

$$\int_0^2 \frac{4x+5}{(2x+1)(x+2)}\,dx$$
$$\int_0^2 \left(\frac{2}{2x+1}+\frac{1}{x+2}\right)dx$$
$$= [\ln|2x+1| + \ln|x+2|]_0^2$$
$$= \ln 5 + \ln 4 - 0 - \ln 2$$
$$= \ln 5 + \ln 2$$
$$= \ln 10$$

h $$\frac{1-2x}{(2x+1)(x+1)^2} = \frac{A}{2x+1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$
$$1-2x = A(x+1)^2 + B(2x+1)(x+1) + C(2x+1)$$
Letting $x=-1$:
$$3 = 0+0-C$$
$$C=-3$$
Letting $x=-\frac{1}{2}$:
$$2 = \frac{1}{4}A + 0 + 0$$
$$A = 8$$
Equating coefficients of x^2:
$$0 = A+2B = 8+2B$$
$$B=-4$$
$$\frac{1-2x}{(2x+1)(x+1)^2} = \frac{8}{2x+1} - \frac{4}{x+1} - \frac{3}{(x+1)^2}$$
$$\int_0^1 \frac{1-2x}{(2x+1)(x+1)^2}\,dx$$
$$\int_0^1 \left(\frac{8}{2x+1} - \frac{4}{x+1} - \frac{3}{(x+1)^2}\right)dx$$
$$= \int_0^1 \left(\frac{8}{2x+1} - \frac{4}{x+1} - 3(x+1)^{-2}\right)dx$$
$$= \left[4\ln|2x+1| - 4\ln|x+1| + 3(x+1)^{-1}\right]_0^1$$
$$= 4\ln 3 - 4\ln 2 + \frac{3}{2} - 4\ln 1 + 4\ln 1 - 3$$
$$= 4\ln\frac{3}{2} - \frac{3}{2}$$
$$= 2\ln\frac{9}{4} - \frac{3}{2}$$

3 **a** $(x-1)(x+2) = x^2+x-2$

$$\begin{array}{r} 1 \\ x^2+x-2\overline{\smash{)}x^2+2x+3} \\ \underline{x^2+x-2} \\ x+5 \end{array}$$

$$\frac{x^2+2x+3}{(x+2)(x-1)} = 1 + \frac{x+5}{(x+2)(x-1)}$$
Splitting the proper fraction $\frac{x+5}{(x+2)(x-1)}$ into partial fractions:
$$\frac{x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$
$$x+5 = A(x-1) + B(x+2)$$
Letting $x=1$:
$$6 = 0+3B$$
$$B=2$$
Letting $x=-2$:

$3 = -3A$
$A = -1$
$\frac{x^2+2x+3}{(x+2)(x-1)} = 1 - \frac{1}{x+2} + \frac{2}{x-1}$
$\int_2^3 \frac{x^2+2x+3}{(x+2)(x-1)}\,\mathrm{d}x$
$= \int_2^3 \left(1 - \frac{1}{x+2} + \frac{2}{x-1}\right)\mathrm{d}x$
$= [x - \ln|x+2| + 2\ln|x-1|]_2^3$
$= 3 - \ln 5 + 2\ln 2 - 2 + \ln 4 - 2\ln 1$
$= 1 - \ln 5 + \ln 4 + \ln 4$
$= 1 + \ln\frac{16}{5}$

b $(x+1)(x+2) = x^2+3x+2$

$$\begin{array}{r} 2 \\ x^2+3x+2\overline{)2x^2+5x+1} \\ \underline{2x^2+6x+4} \\ -x-3 \end{array}$$

$\frac{x^2+2x+3}{(x+2)(x+1)} = 2 - \frac{x+3}{(x+2)(x+1)}$

Splitting the proper fraction $\frac{x+3}{(x+2)(x+1)}$ into partial fractions:

$\frac{x+3}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$
$x+3 = A(x+1) + B(x+2)$

Letting $x = -1$:

$2 = 0 + B$
$B = 2$

Letting $x = -2$:

$1 = -A$
$A = -1$
$\frac{x^2+2x+3}{(x+2)(x-1)} = 2 + \frac{1}{x+2} - \frac{2}{x-1}$
$\int_1^3 \frac{2x^2+5x+1}{(x+2)(x+1)}\,\mathrm{d}x$
$= \int_1^3 \left(2 + \frac{1}{x+2} - \frac{2}{x+1}\right)\mathrm{d}x$
$= [2x + \ln|x+2| - 2\ln|x+1|]_1^3$
$= 6 + \ln 5 - 2\ln 4 - 2 - \ln 3 + 2\ln 2$
$= 6 + \ln 5 - \ln 16 - 2 - \ln 3 + \ln 4$
$= 4 - \ln\frac{12}{5}$

c $(x+1)(2x-1) = 2x^2+x-1$

$$\begin{array}{r} 2 \\ 2x^2+x-1\overline{)4x^2+2x-5} \\ \underline{4x^2+2x-2} \\ -3 \end{array}$$

$\frac{4x^2+2x-5}{(x+1)(2x-1)} = 2 - \frac{3}{(x+1)(2x-1)}$

Splitting the proper fraction $\frac{3}{(x+1)(2x-1)}$ into partial fractions:

$$\frac{3}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1}$$
$$3 = A(2x-1) + B(x+1)$$

Letting $x = -1$:

$3 \ = 0 - 3A$
$A = -1$

Letting $x = \frac{1}{2}$:

$3 \ = \frac{3}{2}B$
$B = 2$

$$\frac{3}{(x+1)(2x-1)} = -\frac{1}{x+1} + \frac{2}{2x-1}$$
$$\int_1^2 \frac{4x^2+2x-5}{(x+1)(2x-1)}\,\mathrm{d}x$$
$$= \int_1^2 \left(2 + \frac{1}{x+1} - \frac{2}{2x-1}\right)\mathrm{d}x$$
$$= [2x + \ln|x+1| - \ln|2x-1|]_1^2$$
$$= 4 + \ln 3 - \ln 3 - 2 - \ln 2 + \ln 1$$
$$= 2 - \ln 2$$

4

When asked to integrate a fraction, check to see if the denominator factorises. In worked solution **4** the difference of two squares is used.

$$\frac{x+3\sqrt{2}}{(x^2-2)} = \frac{x+3\sqrt{2}}{(x-\sqrt{2})(x+\sqrt{2})} = \frac{A}{x-\sqrt{2}} + \frac{B}{x+\sqrt{2}}$$
$$x + 3\sqrt{2} = A(x+\sqrt{2}) + B(x-\sqrt{2})$$

Letting $x = \sqrt{2}$:

$4\sqrt{2} = 2\sqrt{2}A$
$A = 2$

Letting $x = -\sqrt{2}$:

$2\sqrt{2} = -2\sqrt{2}B$
$B = -1$

$$\frac{x+3\sqrt{2}}{(x^2-2)} = \frac{x+3\sqrt{2}}{(x-\sqrt{2})(x+\sqrt{2})} = \frac{2}{x-\sqrt{2}} - \frac{1}{x+\sqrt{2}}$$
$$\int_{2\sqrt{2}}^{3\sqrt{2}} \frac{x+3\sqrt{2}}{(x^2-2)}\,\mathrm{d}x$$
$$\int_{2\sqrt{2}}^{3\sqrt{2}} \left(\frac{2}{x-\sqrt{2}} - \frac{1}{x+\sqrt{2}}\right)\mathrm{d}x$$
$$= [2\ln|x-\sqrt{2}| - \ln|x+\sqrt{2}|]_{2\sqrt{2}}^{3\sqrt{2}}$$
$$= 2\ln(2\sqrt{2}) - \ln(4\sqrt{2}) - 2\ln(\sqrt{2}) + \ln(3\sqrt{2})$$
$$= \ln\frac{24\sqrt{2}}{8\sqrt{2}}$$
$$= \ln 3$$

5 $u = \sin x$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$$
$$\mathrm{d}u = \cos x\,\mathrm{d}x$$

$x = 0 \Rightarrow u = 0$
$x = \frac{\pi}{2} \Rightarrow u = 1$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{9-\sin^2 x}\,\mathrm{d}x$$
$$= \int_0^1 \frac{1}{9-u^2}\,\mathrm{d}u$$
$$= \int_0^1 \frac{1}{(3-u)(3+u)}\,\mathrm{d}u$$
$$\frac{1}{(3-u)(3+u)} = \frac{A}{3-u} + \frac{B}{3+u}$$
$$1 = A(3+u) + B(3-u)$$

Letting $u = 3$:

$$1 = 6A$$
$$A = \frac{1}{6}$$

Letting $u = -3$:

$$1 = 6B$$
$$B = \frac{1}{6}$$

$$\int_0^1 \frac{1}{(3-u)(3+u)}\,\mathrm{d}u$$
$$= \int_0^1 \left(\frac{1}{6(3-u)} + \frac{1}{6(3+u)}\right)\mathrm{d}u$$
$$= \left[-\frac{1}{6}\ln|3-u| + \frac{1}{6}\ln|3+u|\right]_0^1$$
$$= -\frac{1}{6}\ln 2 + \frac{1}{6}\ln 4 + \frac{1}{6}\ln 3 - \frac{1}{6}\ln 3$$
$$= \frac{1}{6}\ln\frac{4}{2}$$
$$= \frac{1}{6}\ln 2$$

6 $u = \mathrm{e}^x$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$$
$$\mathrm{d}u = \mathrm{e}^x\mathrm{d}x$$
$$\mathrm{d}x = \frac{\mathrm{d}u}{u}$$
$$x = 0 \Rightarrow u = 1$$
$$x = \ln 2 \Rightarrow u = 2$$

$$\int_0^{\ln 2} \frac{\mathrm{e}^{2x}}{(1+\mathrm{e}^x)(2\mathrm{e}^x+1)}\,\mathrm{d}x$$
$$= \int_1^2 \frac{u^2}{(1+u)(2u+1)}\frac{\mathrm{d}u}{u}$$
$$= \int_1^2 \frac{u}{(1+u)(2u+1)}\,\mathrm{d}u$$
$$\frac{u}{(1+u)(2u+1)} = \frac{A}{1+u} + \frac{B}{2u+1}$$
$$u = A(2u+1) + B(1+u)$$

Letting $u = -1$:

$$-1 = -A$$
$$A = 1$$

Letting $u = -\frac{1}{2}$:

$$-\frac{1}{2} = \frac{1}{2}B$$
$$B = -1$$

$$
\begin{aligned}
\int_1^2 \frac{u}{(1+u)(2u+1)}\,\mathrm{d}u &= \int_1^2 \left(\frac{1}{1+u} - \frac{1}{2u+1}\right)\mathrm{d}u \\
&= \left[\ln|1+u| - \frac{1}{2}\ln|2u+1|\right]_1^2 \\
&= \ln 3 - \frac{1}{2}\ln 5 - \ln 2 + \frac{1}{2}\ln 3 \\
&= \frac{1}{2}\ln 9 - \frac{1}{2}\ln 5 - \frac{1}{2}\ln 4 + \frac{1}{2}\ln 3 \\
&= \frac{1}{2}\ln\frac{27}{20}
\end{aligned}
$$

EXERCISE 8F

1

Remember that when a logarithm is involved in an integration by parts, it is usually best to choose the logarithmic part as u. This is because it differentiates to give a power of x.

c $u = \ln 2x$ so $\dfrac{du}{dx} = \dfrac{2}{2x} = \dfrac{1}{x}$

$\dfrac{dv}{dx} = x$ so $v = \dfrac{1}{2}x^2$

$$\int x \ln 2x \, dx$$
$$= uv - \int v \frac{du}{dx} \, dx$$
$$= \frac{1}{2}x^2 \ln 2x - \int \frac{1}{2}x^2 \left(\frac{1}{x}\right) dx$$
$$= \frac{1}{2}x^2 \ln 2x - \int \frac{1}{2}x \, dx$$
$$= \frac{1}{2}x^2 \ln 2x - \frac{1}{4}x^2 + c$$

d $u = x$ so $\dfrac{du}{dx} = 1$

$\dfrac{dv}{dx} = \sin 2x$ so $v = -\dfrac{1}{2}\cos 2x$

$$\int x \sin 2x \, dx$$
$$= -\frac{1}{2}x \cos 2x - \int -\frac{1}{2}\cos 2x \, dx$$
$$= -\frac{1}{2}x \cos 2x + \frac{1}{2}\int \cos 2x \, dx$$
$$= -\frac{1}{2}x \cos 2x + \frac{1}{4}\sin 2x + c$$

2 **a** $u = x$ so $\dfrac{du}{dx} = 1$

$\dfrac{dv}{dx} = \cos 3x$ so $v = \dfrac{1}{3}\sin 3x$

$$I = \int_0^{\frac{\pi}{6}} x \cos 3x \, dx$$
$$= \left[\frac{1}{3}x \sin 3x\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \frac{1}{3}\sin 3x \, dx$$
$$= \frac{\pi}{18} - \left[-\frac{1}{9}\cos 3x\right]_0^{\frac{\pi}{6}}$$
$$= \frac{\pi}{18} - \frac{1}{9}$$
$$= \frac{1}{18}(\pi - 2)$$

Note that you need to apply the limits to any expression that is written outside of the integral. For example, in worked solution **2 a**, the expression for 'uv', $\dfrac{1}{3}x \sin 3x$, which forms part of the integration by parts formula, is placed in square brackets with limits.

d

Sometimes you can use integration by parts when there is only one function inside the integral. Just multiply by 1 as shown in worked solution **2 d**.

$$I = \int_1^3 \ln x \, dx = \int_1^3 1 \times \ln x \, dx$$

$u = \ln x$ so $\dfrac{du}{dx} = \dfrac{1}{x}$

$\dfrac{dv}{dx} = 1$ so $v = x$

$$\begin{aligned} I &= [x \ln x]_1^3 - \int_1^3 x \frac{1}{x}\, dx \\ &= 3\ln 3 - \ln 1 - [x]_1^3 \\ &= 3\ln 3 - 3 + 1 \\ &= 3\ln 3 - 2 \\ &= \ln 27 - 2 \end{aligned}$$

3 **a** $$\begin{aligned} I &= \int_1^2 (\ln x)^2\, dx \\ &= \int_1^2 1 \times (\ln x)^2\, dx \end{aligned}$$

$u = (\ln x)^2$ so $\dfrac{du}{dx} = \dfrac{2\ln x}{x}$

$\dfrac{dv}{dx} = 1$ so $v = x$

$$\begin{aligned} &\int_1^2 1 \times (\ln x)^2\, dx \\ &= \left[x(\ln x)^2\right]_1^2 - \int_1^2 2\ln x\, dx \\ &= 2(\ln 2)^2 - 0 - \int_1^2 2\ln x\, dx \end{aligned}$$

Using integration by parts again on $\int_1^2 2\ln x\, dx$:

$u = \ln x$ so $\dfrac{du}{dx} = \dfrac{1}{x}$

$\dfrac{dv}{dx} = 2$ so $v = 2x$

$$\begin{aligned} I &= 2(\ln 2)^2 - \left\{[2x\ln x]_1^2 - \int_1^2 2\, dx\right\} \\ &= 2(\ln 2)^2 - 4\ln 2 + 0 + [2x]_1^2 \\ &= 2(\ln 2)^2 - 4\ln 2 + 4 - 2 \\ &= 2(\ln 2)^2 - 4\ln 2 + 2 \\ &= 2(\ln 2 - 1)^2 \end{aligned}$$

f $$\begin{aligned} I &= [-e^x \cos x]_0^\pi - \int_0^\pi -e^x \cos x\, dx \\ &= -e^\pi(-1) + 1 + \int_0^\pi e^x \cos x\, dx \end{aligned}$$

$I = \int_0^\pi e^x \sin x\, dx$

$u = e^x$ so $\dfrac{du}{dx} = e^x$

$\dfrac{dv}{dx} = \sin x$ so $v = -\cos x$

Using integration by parts again on $\int_0^\pi e^x \cos x\, dx$:

$u = e^x$ so $\dfrac{du}{dx} = e^x$

$\dfrac{dv}{dx} = \cos x$ so $v = \sin x$

$$I = e^{\pi} + 1 + [e^{x} \ \sin x]_0^{\pi} - \int_0^{\pi} e^{x} \ \sin x \, dx$$

$$I = e^{\pi} + 1 + 0 - 0 - I$$

$$2I = e^{\pi} + 1$$

$$I = \frac{e^{\pi} + 1}{2}$$

Two things are really important in worked solution **3 f**. Firstly, if you use $u = e^{x}$ then you must do the same again in the second integral. Try swapping at this stage and you will just get back to where you started.

Secondly, it is important that you call the original integral something like 'I', so that you can solve the equation later.

4 **a** $y = x^2 \ln x$

$y = 0$ when $x = 0$ or $\ln x = 0$

i.e. when $x = 0$ or $x = 1$

$$\text{Area} = \int_1^{e} x^2 \ln x \, dx$$

$$= \left[\frac{1}{3}x^3 \ln x\right]_1^{e} - \int_1^{e} \frac{1}{3}x^3 \left(\frac{1}{x}\right) dx$$

$$= \frac{1}{3}e^3 - \int_1^{e} \frac{1}{3}x^2 \, dx$$

$$= \frac{1}{3}e^3 - \left[\frac{1}{9}x^3\right]_1^{e}$$

$$= \frac{1}{3}e^3 - \frac{1}{9}e^3 + \frac{1}{9}$$

$$= \frac{2}{9}e^3 + \frac{1}{9}$$

$$= \frac{1}{9}(2e^3 + 1)$$

b $\int_0^{\pi} x \sin x \, dx$

$$= [-x \cos x]_0^{\pi} - \int_0^{\pi} - \cos x \, dx$$

$$= \pi + 0 + [\sin x]_0^{\pi}$$

$$= \pi + 0$$

$$= \pi$$

5 $y = 0$ when $x = -2$

$$V = \int_{-2}^{0} \pi y^2 \, dx$$

$$= \pi \int_{-2}^{0} e^{-2x}(x + 2) \, dx$$

$u = x + 2$ so $\frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-2x}$ so $v = -\frac{1}{2}e^{-2x}$

$$V = \pi\left[-\frac{1}{2}(x + 2)e^{-2x}\right]_{-2}^{0} - \pi \int_{-2}^{0} -\frac{1}{2}e^{-2x} \, dx$$

$$= -\pi + 0 + \frac{1}{2}\pi \int_{-2}^{0} e^{-2x} \, dx$$

$$= -\pi + \frac{1}{2}\pi\left[-\frac{1}{2}e^{-2x}\right]_{-2}^{0}$$

$$= -\pi - \frac{1}{4}\pi + \frac{1}{4}\pi e^4$$

$$= \frac{1}{4}\pi(e^4 - 5)$$

END-OF-CHAPTER REVIEW EXERCISE 8

P3 **This exercise is for Pure Mathematics 3 students only.**

1 Remember that you are likely to need integration by parts if your integrand is a product and neither term in the product is the derivative of the other.

Using integration by parts with $u = x$ and $\frac{dv}{dx} = e^{-2x}$:

$$\int_0^{\frac{1}{2}} x e^{-2x}\, dx$$
$$= \left[-\frac{1}{2}xe^{-2x}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} -\frac{1}{2}e^{-2x}\, dx$$
$$= -\frac{1}{4}e^{-1} + 0 + \frac{1}{2}\int_0^{\frac{1}{2}} e^{-2x}\, dx$$
$$= -\frac{1}{4}e^{-1} + \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_0^{\frac{1}{2}}$$
$$= -\frac{1}{4}e^{-1} + \frac{1}{2}\left(-\frac{1}{2}e^{-1} + \frac{1}{2}\right)$$
$$= \frac{1}{4} - \frac{1}{2}e^{-1}$$
$$= \frac{1}{4}\left(1 - \frac{2}{e}\right)$$

2 Using integration by parts with $u = \ln x$ and $\frac{dv}{dx} = x^{-\frac{1}{2}}$:

$$\int_1^4 x^{-\frac{1}{2}} \ln x\, dx$$
$$= \left[2x^{\frac{1}{2}} \ln x\right]_1^4 - \int_1^4 \frac{2x^{\frac{1}{2}}}{x}\, dx$$
$$= 4\ln 4 - 0 - \int_1^4 2x^{-\frac{1}{2}}\, dx$$
$$= 4\ln 4 - \left[4x^{\frac{1}{2}}\right]_1^4$$
$$= 4\ln 4 - 8 + 4$$
$$= 4\ln 4 - 4$$
$$= 4(\ln 4 - 1)$$

3 $u = 1 + 3\tan x$

$\frac{du}{dx} = 3\sec^2 x$

$du = 3\sec^2 x\, dx$

$x = 0 \Rightarrow u = 1$

$x = \frac{\pi}{4} \Rightarrow u = 4$

$$\int_0^{\frac{\pi}{4}} \frac{\sqrt{1+3\tan x}}{\cos^2 x}\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \sqrt{1+3\tan x}\ \sec^2 x\,\mathrm{d}x$$
$$= \frac{1}{3}\int_1^4 \sqrt{u}\,\mathrm{d}u$$
$$= \frac{1}{3}\int_1^4 u^{\frac{1}{2}}\,\mathrm{d}u$$
$$= \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^4$$
$$= \frac{1}{3}\left(\frac{16}{3} - \frac{2}{3}\right)$$
$$= \frac{14}{9}$$

4 **i** $\cot\theta + \tan\theta$

$$\equiv \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$
$$\equiv \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$
$$\equiv \frac{1}{\sin\theta\cos\theta}$$
$$\equiv \frac{2}{2\sin\theta\cos\theta}$$
$$\equiv \frac{2}{\sin 2\theta}$$
$$\equiv 2\operatorname{cosec} 2\theta$$

ii Using the results from part **i**:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec} 2\theta\,\mathrm{d}\theta$$
$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\cot\theta + \tan\theta)\,\mathrm{d}\theta$$
$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\right)\mathrm{d}\theta$$
$$= \frac{1}{2}[\ln\sin\theta + \ln\sec\theta]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= \frac{1}{2}\ln\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\ln 2 - \frac{1}{2}\left(\ln\frac{1}{2} + \ln\frac{2}{\sqrt{3}}\right)$$
$$= \frac{1}{2}\ln\left(\frac{\sqrt{3}}{2}\right) + \ln 2 + \frac{1}{2}\ln\left(\frac{\sqrt{3}}{2}\right)$$
$$= \ln\left(\frac{\sqrt{3}}{2}\right) + \ln 2$$
$$= \ln\sqrt{3}$$
$$= \frac{1}{2}\ln 3$$

5 **i** $x = 2\sin\theta$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta$$
$$x = 0 \Rightarrow \theta = 0$$
$$x = 1 \Rightarrow \theta = \frac{\pi}{6}$$

$$
\begin{aligned}
I &= \int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} 2\cos\theta\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\sqrt{4\cos^2\theta}} 2\cos\theta\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{6}} 4\sin^2\theta\,\mathrm{d}\theta
\end{aligned}
$$

ii Using the result from part **i**:

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{6}} 4\sin^2\theta\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{6}} 4\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{6}} (2-2\cos 2\theta)\,\mathrm{d}\theta \\
&= [2\theta - \sin 2\theta]_0^{\frac{\pi}{6}} \\
&= \frac{\pi}{3} - \frac{\sqrt{3}}{2} - 0 + 0 \\
&= \frac{1}{6}(2\pi - 3\sqrt{3})
\end{aligned}
$$

6 **i**

$$
\begin{aligned}
x &= t^2+1 \\
t^2 &= x-1 \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= 2t \\
\mathrm{d}x &= 2t\,\mathrm{d}t \\
2t^3\,\mathrm{d}t &= t^2\,\mathrm{d}x = (x-1)\,\mathrm{d}x \\
t &= 0 \Rightarrow x = 1 \\
t &= 2 \Rightarrow x = 5
\end{aligned}
$$

$$
\begin{aligned}
I &= \int_0^2 4t^3\ln(t^2+1)\,\mathrm{d}t \\
&= \int_1^5 2(x-1)\ln x\,\mathrm{d}x
\end{aligned}
$$

ii Using the results from part **i** and integration by parts:

$$
\begin{aligned}
I &= \int_1^5 2(x-1)\ln x\,\mathrm{d}x \\
&= \left[(x-1)^2\ln x\right]_1^5 - \int_1^5 \frac{(x-1)^2}{x}\,\mathrm{d}x \\
&= 16\ln 5 - 0 - \int_1^5 \frac{x^2-2x+1}{x}\,\mathrm{d}x \\
&= 16\ln 5 - \int_1^5 \left(x-2+\frac{1}{x}\right)\mathrm{d}x \\
&= 16\ln 5 - \left[\frac{1}{2}x^2 - 2x + \ln x\right]_1^5 \\
&= 16\ln 5 - \frac{25}{2} + 10 - \ln 5 + \frac{1}{2} - 2 + 0 \\
&= 15\ln 5 - 4
\end{aligned}
$$

7 **i**

$$
\begin{aligned}
\frac{2}{(x+1)(x+3)} &= \frac{A}{x+1} + \frac{B}{x+3} \\
2 &= A(x+3) + B(x+1)
\end{aligned}
$$

Letting $x = -1$:

$$
\begin{aligned}
2 &= 2A \\
A &= 1
\end{aligned}
$$

Letting $x = -3$:

$$2 = -2B$$
$$B = -1$$
$$\frac{2}{(x+1)(x+3)} = \frac{1}{x+1} - \frac{1}{x+3}$$

ii Using the result from part **i**:

$$\left(\frac{2}{(x+1)(x+3)}\right)^2 = \left(\frac{1}{x+1} - \frac{1}{x+3}\right)^2$$
$$= \frac{1}{(x+1)^2} - \frac{2}{(x+1)(x+3)} + \frac{1}{(x+3)^2}$$
$$= \frac{1}{(x+1)^2} - \left(\frac{1}{x+1} - \frac{1}{x+3}\right) + \frac{1}{(x+3)^2}$$
$$= \frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x+3} + \frac{1}{(x+3)^2}$$

iii

> Be careful to separate functions involving powers of -1 from the rest. The functions involving powers of -1 will be integrated using logarithms.

From the results in parts **i** and **ii**:

$$\int_0^1 \frac{4}{(x+1)^2(x+3)^2}\,dx$$
$$= \int_0^1 \left[\frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x+3} + \frac{1}{(x+3)^2}\right] dx$$
$$= \int_0^1 \left[(x+1)^{-2} - \frac{1}{x+1} + \frac{1}{x+3} + (x+3)^{-2}\right] dx$$
$$= \left[-(x+1)^{-1} - \ln|x+1| + \ln|x+3| - (x+3)^{-1}\right]_0^1$$
$$= -\frac{1}{2} - \ln 2 + \ln 4 - \frac{1}{4} + 1 + \ln 1 - \ln 3 + \frac{1}{3}$$
$$= -\frac{1}{2} - \frac{1}{4} + 1 + \frac{1}{3} + \ln\frac{4}{6}$$
$$= \frac{7}{12} + \ln\frac{2}{3}$$
$$= \frac{7}{12} - \ln\frac{3}{2}$$

8 **a** $\int (4 + \tan^2 2x)\,dx$

$$= \int (4 + \sec^2 2x - 1)\,dx$$
$$= \int (3 + \sec^2 2x)\,dx$$
$$= 3x + \frac{1}{2}\tan 2x + c$$

b $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin\left(x + \frac{\pi}{6}\right)}{\sin x}\,dx$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6}}{\sin x}\,dx$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{\sqrt{3}}{2} + \frac{\cos x}{2\sin x}\right) dx$$
$$= \left[\frac{\sqrt{3}}{2}x + \frac{1}{2}\ln \sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= \frac{\sqrt{3}}{4}\pi + \frac{1}{2}\ln 1 - \frac{\sqrt{3}}{8}\pi - \frac{1}{2}\ln\frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{3}}{8}\pi - \frac{1}{2}\ln\frac{\sqrt{2}}{2}$$

9 **i** $\tan 2\theta - \tan\theta$

$$\equiv \frac{\sin 2\theta}{\cos 2\theta} - \frac{\sin\theta}{\cos\theta}$$
$$\equiv \frac{2\sin\theta\cos\theta}{\cos 2\theta} - \frac{\sin\theta}{\cos\theta}$$
$$\equiv \frac{2\sin\theta\cos^2\theta}{\cos\theta\cos 2\theta} - \frac{\sin\theta\cos 2\theta}{\cos\theta\cos 2\theta}$$
$$\equiv \frac{2\sin\theta\cos^2\theta - \sin\theta\cos 2\theta}{\cos\theta\cos 2\theta}$$
$$\equiv \frac{2\sin\theta\cos^2\theta - \sin\theta(2\cos^2\theta - 1)}{\cos\theta\cos 2\theta}$$
$$\equiv \frac{\sin\theta}{\cos\theta}\sec 2\theta$$
$$\equiv \tan\theta\sec 2\theta$$

ii From the results in part **i**:

$$\int_0^{\frac{\pi}{6}} \tan\theta\sec 2\theta\,\mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{6}} (\tan 2\theta - \tan\theta)\,\mathrm{d}\theta$$
$$= \left[\frac{1}{2}\ln\sec 2\theta - \ln\sec\theta\right]_0^{\frac{\pi}{6}}$$
$$= \frac{1}{2}\ln 2 - \ln\frac{2}{\sqrt{3}} - 0 + 0$$
$$= \frac{1}{2}\ln 2 - \frac{1}{2}\ln\frac{4}{3}$$
$$= \frac{1}{2}\ln\frac{3}{2}$$

10 **i**

Although you will find more than one solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, the diagram in the question shows that you only need one solution.

Using the product rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (2 - 2x)\mathrm{e}^{\frac{1}{2}x} + \frac{1}{2}(2x - x^2)\mathrm{e}^{\frac{1}{2}x}$$
$$= 2\mathrm{e}^{\frac{1}{2}x} - 2x\mathrm{e}^{\frac{1}{2}x} + x\mathrm{e}^{\frac{1}{2}x} - \frac{1}{2}x^2\mathrm{e}^{\frac{1}{2}x}$$
$$= \mathrm{e}^{\frac{1}{2}x}\left(2 - 2x + x - \frac{1}{2}x^2\right)$$

Using the fact that M is a maximum point so at this point $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$2 - 2x + x - \frac{1}{2}x^2 = 0$$
$$x^2 + 2x - 4 = 0$$
$$(x + 1)^2 - 5 = 0$$
$$x = -1 \pm \sqrt{5}$$

For M, $x > 0$

so $x = -1 + \sqrt{5}$

ii Using integration by parts with $u = (2x - x^2)$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{\frac{1}{2}x}$:

$$I = \int_0^2 (2x - x^2)\mathrm{e}^{\frac{1}{2}x}\,\mathrm{d}x$$
$$= \left[2(2x - x^2)\mathrm{e}^{\frac{1}{2}x}\right]_0^2 - \int_0^2 2(2 - 2x)\mathrm{e}^{\frac{1}{2}x}\,\mathrm{d}x$$
$$= 0 - 0 - 4\int_0^2 (1 - x)\mathrm{e}^{\frac{1}{2}x}\,\mathrm{d}x$$

Using integration by parts again on $4\int_0^2 (1-x)e^{\frac{1}{2}x}\,dx$:

$$\begin{aligned} I &= -4\left[2e^{\frac{1}{2}x}(1-x)\right]_0^2 + 4\int_0^2 2e^{\frac{1}{2}x}(-1)\,dx \\ &= 8e + 8 - 8\left[2e^{\frac{1}{2}x}\right]_0^2 \\ &= 8e + 8 - 16e + 16 \\ &= 24 - 8e \end{aligned}$$

11

$$\begin{array}{r} 2 \\ 2x^2+5x+2\overline{)4x^2+5x+3} \\ \underline{4x^2+10x+4} \\ -5x-1 \end{array}$$

$$\begin{aligned} \frac{4x^2+5x+3}{2x^2+5x+2} &= 2 - \frac{5x+1}{2x^2+5x+2} \\ &= 2 - \frac{5x+1}{(2x+1)(x+2)} \end{aligned}$$

Splitting the proper fraction $\frac{5x+1}{(2x+1)(x+2)}$ into partial fractions:

$$\begin{aligned} \frac{5x+1}{(2x+1)(x+2)} &= \frac{A}{2x+1} + \frac{B}{x+2} \\ 5x+1 &= A(x+2) + B(2x+1) \end{aligned}$$

Letting $x = -2$:

$$\begin{aligned} -9 &= -3B \\ B &= 3 \end{aligned}$$

Letting $x = -\frac{1}{2}$:

$$\begin{aligned} -\frac{3}{2} &= \frac{3}{2}A \\ A &= -1 \end{aligned}$$

$$\frac{4x^2+5x+3}{2x^2+5x+2} = 2 + \frac{1}{2x+1} - \frac{3}{x+2}$$

So $\int_0^4 \frac{4x^2+5x+3}{2x^2+5x+2}\,dx$

$$\begin{aligned} &\int_0^4 \left(2 + \frac{1}{2x+1} - \frac{3}{x+2}\right) dx \\ &= \left[2x + \frac{1}{2}\ln|2x+1| - 3\ln|x+2|\right]_0^4 \\ &= 8 + \frac{1}{2}\ln 9 - 3\ln 6 + 3\ln 2 \\ &= 8 + \ln 3 + 3\ln\frac{1}{3} \\ &= 8 + \ln 3 - 3\ln 3 \\ &= 8 - 2\ln 3 \\ &= 8 - \ln 9 \end{aligned}$$

12 **i**

Although you will find more than one solution to $\frac{dy}{dx} = 0$, the diagram in the question shows that you only need one solution.

Using the product rule:

$$\frac{dy}{dx} = 3x^2\ln x + \frac{x^3}{x} = 3x^2\ln x + x^2$$

M is a minimum point, so at M, $3x^2\ln x + x^2 = 0$:

$$x^2(3\ln x + 1) = 0$$

$$x = 0 \text{ or } \ln x = -\frac{1}{3}$$

But from the diagram, $x > 0$

so $x = \mathrm{e}^{-\frac{1}{3}}$

$y = \mathrm{e}^{-1} \ln \mathrm{e}^{-\frac{1}{3}} = -\frac{1}{3\mathrm{e}}$

The coordinates of M are $\left(\mathrm{e}^{-\frac{1}{3}}, -\frac{1}{3\mathrm{e}}\right)$.

ii When $y = 0$:

$x^3 \ln x = 0$

$x = 0$ or $x = 1$

$\text{Area} = \int_1^2 x^3 \ln x \, \mathrm{d}x$

Using integration by parts:

$$\begin{aligned}\text{Area} &= \left[\frac{1}{4}x^4 \ln x\right]_1^2 - \int_1^2 \frac{1}{4}\frac{x^4}{x}\,\mathrm{d}x \\ &= 4\ln 2 - 0 - \frac{1}{4}\int_1^2 x^3\,\mathrm{d}x \\ &= 4\ln 2 - \frac{1}{4}\left[\frac{1}{4}x^4\right]_1^2 \\ &= 4\ln 2 - \frac{1}{4}\left(4 - \frac{1}{4}\right) \\ &= 4\ln 2 - \frac{15}{16}\end{aligned}$$

13 **i** Using the quotient rule:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x(1+x^3) - x^2(3x^2)}{(1+x^3)^2} \\ &= \frac{2x + 2x^4 - 3x^4}{(1+x^3)^2} \\ &= \frac{2x - x^4}{(1+x^3)^2}\end{aligned}$$

Since M is a maximum point at M, $\frac{2x - x^4}{(1+x^3)^2} = 0$:

$$\begin{aligned}2x - x^4 &= 0 \\ x(2 - x^3) &= 0\end{aligned}$$

$x = 0$ or $x^3 = 2$

From the diagram $x > 0$

so $x = \sqrt[3]{2}$

ii

$$\begin{aligned}\int_1^p \frac{x^2}{1+x^3}\,\mathrm{d}x &= 1 \\ \frac{1}{3}\int_1^p \frac{3x^2}{1+x^3}\,\mathrm{d}x &= 1 \\ \frac{1}{3}\left[\ln|1+x^3|\right]_1^p &= 1 \\ \ln(1+p^3) - \ln 2 &= 3 \\ \ln(1+p^3) &= 3 + \ln 2 \\ 1 + p^3 &= \mathrm{e}^{3+\ln 2} = \mathrm{e}^{\ln 2}\mathrm{e}^3 = 2\mathrm{e}^3 \\ p^3 &= 2\mathrm{e}^3 - 1 \\ p &= \sqrt[3]{2\mathrm{e}^3 - 1} = 3.40 \text{ (3 s.f.)}\end{aligned}$$

CROSS-TOPIC REVIEW EXERCISE 3

P3 **This exercise is for Pure Mathematics 3 students only.**

1 $(1-3x)^{-5}$

$= 1 + (-5)(-3x) + \frac{(-5)(-6)}{2!}(-3x)^2 + \frac{(-5)(-6)(-7)}{3!}(-3x)^3 + \dots$

$= 1 + 15x + 135x^2 + 945x^3 + \dots$

2

Given that a is a complicated fraction in this question, keep it as a until the end.

Use the fact that $\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$.

$\int_0^1 \frac{1}{3x^2+1}\,\mathrm{d}x$

$= \frac{1}{3}\int_0^1 \frac{1}{x^2+\frac{1}{3}}\,\mathrm{d}x$

$a^2 = \frac{1}{3}$

$$\int_0^1 \frac{1}{3x^2+1}\,\mathrm{d}x = \frac{1}{3}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]_0^1$$

$$= \frac{1}{3}\left[\sqrt{3}\tan^{-1}(x\sqrt{3})\right]_0^1$$

$$= \frac{\sqrt{3}\tan^{-1}\sqrt{3} - 0}{3}$$

$$= \frac{\sqrt{3}}{3}\left(\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{9}\pi$$

3 $32(x+2)^{-3}$

$= 32\left(2^{-3}\right)\left(1+\frac{x}{2}\right)^{-3}$

$= 4\left\{1 + (-3)\left(\frac{x}{2}\right) + \frac{(-3)(-4)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-3)(-4)(-5)}{3!}\left(\frac{x}{2}\right)^3 + \dots\right\}$

$= 4 - 6x + 6x^2 - 5x^3 + \dots$

4 $(4-2x)^{-\frac{1}{2}}$

$= 4^{-\frac{1}{2}}\left(1-\frac{x}{2}\right)^{-\frac{1}{2}}$

$= \frac{1}{2}\left\{1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{2}\right)^2 + \dots\right\}$

$= \frac{1}{2} + \frac{1}{8}x + \frac{3}{64}x^2 + \dots$

Check the question carefully! This question only asks you to expand up to and including the term in x^2, whereas Questions **1** and **3** required one term more. Don't waste precious time by taking the expansion any further than you need to.

5 Using integration by parts:

$$\int_0^{\frac{\pi}{2}} x^2 \sin x \,\mathrm{d}x$$
$$= \left[x^2(-\cos x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -2x\cos x \,\mathrm{d}x$$
$$= 0 - 0 + 2\int_0^{\frac{\pi}{2}} x\cos x \,\mathrm{d}x$$
$$= 2[x\sin x]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} \sin x \,\mathrm{d}x$$
$$= 2\left(\frac{\pi}{2} - 0\right) - 2[-\cos x]_0^{\frac{\pi}{2}}$$
$$= \pi + 2(0-1)$$
$$= \pi - 2$$

6 $\sqrt{\dfrac{1+2x}{1-x}} = (1+2x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$

$(1+2x)^{\frac{1}{2}}$

$$= 1 + \left(\frac{1}{2}\right)(2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(2x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(2x)^3 + \ldots$$
$$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \ldots$$

$(1-x)^{-\frac{1}{2}}$

$$= 1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x)^3 + \ldots$$
$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots$$

$$(1+2x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}} = \left(1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \ldots\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots\right)$$
$$= 1 + \left(\frac{1}{2} + 1\right)x + \left(\frac{3}{8} - \frac{1}{2} + \frac{1}{2}\right)x^2 + \left(\frac{5}{16} + \frac{1}{2} + \frac{3}{8} - \frac{1}{4}\right)x^3 + \ldots$$
$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2 + \frac{15}{16}x^3 + \ldots$$

7 Using integration by parts:

$$\int_0^1 (x+2)\,\mathrm{e}^{-2x} \,\mathrm{d}x$$
$$= \left[-\frac{1}{2}(x+2)\mathrm{e}^{-2x}\right]_0^1 - \int_0^1 -\frac{1}{2}\mathrm{e}^{-2x} \,\mathrm{d}x$$
$$= -\frac{3}{2}\mathrm{e}^{-2} + 1 + \frac{1}{2}\int_0^1 \mathrm{e}^{-2x} \,\mathrm{d}x$$
$$= 1 - \frac{3}{2\mathrm{e}^2} + \frac{1}{2}\left[-\frac{1}{2}\mathrm{e}^{-2x}\right]_0^1$$
$$= 1 - \frac{3}{2\mathrm{e}^2} - \frac{1}{4}(\mathrm{e}^{-2} - 1)$$
$$= 1 - \frac{3}{2\mathrm{e}^2} - \frac{1}{4\mathrm{e}^2} + \frac{1}{4}$$
$$= \frac{5}{4} - \frac{6}{4\mathrm{e}^2} - \frac{1}{4\mathrm{e}^2}$$
$$= \frac{5}{4} - \frac{7}{4\mathrm{e}^2}$$

8 **i** $(1-4x)^{-\frac{1}{2}}$

$$= 1 + \left(-\frac{1}{2}\right)(-4x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-4x)^2 + \ldots$$
$$= 1 + 2x + 6x^2 + \ldots$$

ii $\frac{1+2x}{\sqrt{4-16x}}$

$= \frac{1+2x}{\sqrt{4}\sqrt{1-4x}}$

$= \frac{1}{2}(1+2x)(1-4x)^{-\frac{1}{2}}$

$= \frac{1}{2}(1+2x)\left(1+2x+6x^2+...\right)$

Coefficient of x^2

$= \frac{1}{2}(6+4)$

$= 5$

Note that you do not need to expand the entire expression. You just need to work out the different ways in which you can form terms in x^2, i.e. $1 \times 6x^2$ and $2x \times 2x$. Don't forget to multiply by the $\frac{1}{2}$.

9 **i** $(1+ax)^{-2}$

$= 1 + (-2)(ax) + \frac{(-2)(-3)}{2!}(ax)^2 + \frac{(-2)(-3)(-4)}{3!}(ax)^3 + ...$

$= 1 - 2ax + 3a^2x^2 - 4a^3x^3 + ...$

Using the fact that the coefficients of x and x^3 are equal:

$$-2a = -4a^3$$

$$4a^3 - 2a = 0$$

$$2a\left(2a^2 - 1\right) = 0$$

$a > 0$

$$2a^2 - 1 = 0$$

$$a^2 = \frac{1}{2}$$

$$a = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

ii $(1+ax)^{-2} = 1 - 2ax + 3a^2x^2 - 4a^3x^3 + ...$

$\left(1 + \frac{\sqrt{2}x}{2}\right)^{-2}$

$= 1 - 2\left(\frac{\sqrt{2}}{2}\right)x + 3\left(\frac{\sqrt{2}}{2}\right)^2 x^2 - ...$

$= 1 - \sqrt{2}x + \frac{3}{2}x^2 - ...$

10 **i** $y = 8\sin\frac{1}{2}x - \tan\frac{1}{2}x$

$\frac{dy}{dx} = 4\cos\frac{1}{2}x - \frac{1}{2}\sec^2\frac{1}{2}x$

At the maximum point, $x = \alpha$ and $\frac{dy}{dx} = 0$:

$$4\cos\frac{1}{2}\alpha-\frac{1}{2}\sec^2\frac{1}{2}\alpha=0$$
$$8\cos\frac{1}{2}\alpha=\sec^2\frac{1}{2}\alpha$$
$$8\cos\frac{1}{2}\alpha=\frac{1}{\cos^2\frac{1}{2}\alpha}$$
$$\cos^3\frac{1}{2}\alpha=\frac{1}{8}$$
$$\cos\frac{1}{2}\alpha=\frac{1}{2}$$
$$\frac{1}{2}\alpha=\frac{\pi}{3}$$
$$\alpha=\frac{2\pi}{3}$$

ii $\int_0^{\alpha}\left(8\sin\frac{1}{2}x-\tan\frac{1}{2}x\right)\mathrm{d}x$

$$=\left[-16\cos\frac{1}{2}x+2\ln\left|\cos\frac{1}{2}x\right|\right]_0^{\alpha}$$
$$=-16\cos\frac{\alpha}{2}+2\ln\left|\cos\frac{1}{2}\alpha\right|+16-2\ln 1$$
$$=-16\cos\frac{\pi}{3}+2\ln\cos\frac{\pi}{3}+16$$
$$=-8+2\ln\frac{1}{2}+16$$
$$=8+2\ln\frac{1}{2}$$

11 $\frac{5-3x}{(x+1)(3x+1)}=\frac{A}{x+1}+\frac{B}{3x+1}$

$$5-3x=A(3x+1)+B(x+1)$$

Letting $x=-1$:

$8=-2A+0$

$A=-4$

Letting $x=-\frac{1}{3}$:

$6=0+\frac{2}{3}B$

$B=9$

$$\frac{5-3x}{(x+1)(3x+1)}=-\frac{4}{x+1}+\frac{9}{3x+1}$$
$$\int_0^5\frac{5-3x}{(x+1)(3x+1)}\,\mathrm{d}x=\int_0^5\left(-\frac{4}{x+1}+\frac{9}{3x+1}\right)\mathrm{d}x$$
$$=[-4\ln|x+1|+3\ln|3x+1|]_0^5$$
$$=-4\ln 6+3\ln 16+4\ln 1-3\ln 1$$
$$=3\ln 16-4\ln 6$$
$$=3\ln 2^4-4\ln 6$$
$$=4(3\ln 2)-4\ln 6$$
$$=4\ln 2^3-4\ln 6$$
$$=4\ln 8-4\ln 6$$
$$=4\ln\frac{8}{6}$$
$$=4\ln\frac{4}{3}$$

12 **a** $x=\sin\theta$

$\frac{\mathrm{d}x}{\mathrm{d}\theta}=\cos\theta$

$\mathrm{d}x=\cos\theta\,\mathrm{d}\theta$

$x=0\Rightarrow\theta=0$

$x=\frac{1}{2}\Rightarrow\sin\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{6}$

$$I = \int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta \, d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\sqrt{\cos^2\theta}}\cos\theta \, d\theta$$
$$= \int_0^{\frac{\pi}{6}} 4\sin^2\theta \, d\theta$$

b $$= \int_0^{\frac{\pi}{6}} 4\sin^2\theta \, d\theta$$
$$= \int_0^{\frac{\pi}{6}} 4\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$
$$= \int_0^{\frac{\pi}{6}} (2 - 2\cos 2\theta) \, d\theta$$
$$= [2\theta - \sin 2\theta]_0^{\frac{\pi}{6}}$$
$$= \frac{\pi}{3} - \sin\frac{\pi}{3} - 0 + 0$$
$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

13 **a** $\int_1^a x \ln x \, dx = 30$

Using integration by parts:

$$\left[\frac{1}{2}x^2 \ln x\right]_1^a - \int_1^a \left(\frac{1}{2}x^2 \times \frac{1}{x}\right) dx = 30$$
$$\frac{1}{2}a^2 \ln a - 0 - \int_1^a \frac{1}{2}x \, dx = 30$$
$$\frac{1}{2}a^2 \ln a - \left[\frac{1}{4}x^2\right]_1^a = 30$$
$$\frac{1}{2}a^2 \ln a - \left(\frac{1}{4}a^2 - \frac{1}{4}\right) = 30$$
$$\frac{1}{2}a^2 \ln a - \frac{1}{4}a^2 = 30 - \frac{1}{4}$$
$$2a^2 \ln a - a^2 = 120 - 1$$
$$a^2(2\ln a - 1) = 119$$
$$a^2 = \frac{119}{2\ln a - 1}$$
$$a > 1$$
$$a = \sqrt{\frac{119}{2\ln a - 1}}$$

b

> You can try various starting points and see what happens. Here, the start value of 5 is used simply because it is a value greater than 1.

Using $a_{n+1} = \sqrt{\frac{119}{2\ln a_n - 1}}$ with $a_0 = 5$:

$a_1 = 7.3233$
$a_2 = 6.3170$
$a_3 = 6.6555$
$a_5 = 6.5298$
$a_6 = 6.5749$
$a_7 = 6.5585$
$a_8 = 6.5644$
$a_9 = 6.5623$
$a_{10} = 6.5631$
$a_{11} = 6.5628$
$a_{12} = 6.5629$

$a = 6.56$ to 2 decimal places.

14 **i** $y = x^2 e^{2-x}$

Using the product rule:

$$\frac{dy}{dx} = 2xe^{2-x} - x^2 e^{2-x}$$
$$= e^{2-x}(2x - x^2)$$

At M, $\frac{dy}{dx} = 0$:

$$2x - x^2 = 0$$
$$x(2 - x) = 0$$
$$x = 0 \text{ or } x = 2$$

$x > 0$ (from the diagram)

$x = 2$

ii The x^2 term means that you need to use integration by parts twice.

Using integration by parts twice:

$$\int_0^2 x^2 e^{2-x}\, dx$$
$$= \left[-x^2 e^{2-x}\right]_0^2 - \int_0^2 -2xe^{2-x}\, dx$$
$$= -4 + 0 + 2\int_0^2 xe^{2-x}\, dx$$
$$= -4 + 2\left[-xe^{2-x}\right]_0^2 - 2\int_0^2 -e^{2-x}\, dx$$
$$= -4 - 4 + 0 + 2\left[-e^{2-x}\right]_0^2$$
$$= -8 + 2(-1 + e^2)$$
$$= 2e^2 - 10$$

15 **i** $f(x) = \frac{12 + 8x - x^2}{(2 - x)(4 + x^2)} = \frac{A}{2 - x} + \frac{Bx + C}{4 + x^2}$

$12 + 8x - x^2 = A(4 + x^2) + (Bx + C)(2 - x)$

Letting $x = 2$:

$$24 = 8A + 0$$
$$A = 3$$

Letting $x = 0$:

$$12 = 4A + 2C$$
$$12 = 12 + C$$
$$C = 0$$

Equating coefficients of x^2:

$$-1 = A - B$$
$$-1 = 3 - B$$
$$B = 4$$

$$f(x) = \frac{12 + 8x - x^2}{(2 - x)(4 + x^2)} = \frac{3}{2 - x} + \frac{4x}{4 + x^2}$$

ii $\int_0^1 \frac{12 + 8x - x^2}{(2 - x)(4 + x^2)} dx = \int_0^1 \left(\frac{3}{2 - x} + \frac{4x}{4 + x^2}\right) dx$

$$= \left[-3\ln|2 - x| + 2\ln|4 + x^2|\right]_0^1$$
$$= -3\ln 1 + 2\ln 5 + 3\ln 2 - 2\ln 4$$
$$= 2\ln 5 + 3\ln 2 - 2\ln 4$$
$$= \ln 25 + \ln 8 - \ln 16$$
$$= \ln\left(\frac{25 \times 8}{16}\right)$$
$$= \ln\left(\frac{25}{2}\right)$$

16 **i** $\int_1^a 1 \times \ln(2x)\,\mathrm{d}x = 1$

Using integration by parts:

$$[x\ln(2x)]_1^a - \int_1^a x \times \frac{1}{x}\,\mathrm{d}x = 1$$

$$a\ln 2a - \ln 2 - \int_1^a 1\,\mathrm{d}x = 1$$

$$a\ln 2a - \ln 2 - [x]_1^a = 1$$

$$a\ln 2a - \ln 2 - a + 1 = 1$$

$$a\ln 2a = \ln 2 + a$$

$$\ln 2a = 1 + \frac{\ln 2}{a}$$

$$2a = \exp\left(1 + \frac{\ln 2}{a}\right)$$

$$a = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a}\right)$$

Note that $\exp(x)$ means the same as e^x.

ii

In this worked solution, a starting value of 3 is used as it is greater than 1. Try some other starting values for yourself and see how quickly they converge.

Using $a_{n+1} = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a_n}\right)$ with $a_0 = 3$:

$a_1 = 1.7124$

$a_2 = 2.0373$

$a_3 = 1.9100$

$a_4 = 1.9538$

$a_5 = 1.9379$

$a_6 = 1.9436$

$a_7 = 1.9416$

$a_8 = 1.9423$

$a_9 = 1.9420$

$a_{10} = 1.9421$

$a_{11} = 1.9421$

$a_{12} = 1.9421$

$a = 1.94$ to 2 decimal places.

17 **i** $\frac{4 + 12x + x^2}{(3-x)(1+2x)^2} = \frac{A}{3-x} + \frac{B}{1+2x} + \frac{C}{(1+2x)^2}$

$$4 + 12x + x^2 = A(1+2x)^2 + B(3-x)(1+2x) + C(3-x)$$

Letting $x = 3$:

$$49 = 49A + 0 + 0$$

$$A = 1$$

Letting $x = -\frac{1}{2}$:

$$-\frac{7}{4} = 0 + 0 + \frac{7}{2}C$$

$$C = -\frac{1}{2}$$

Letting $x = 0$:

$$4 = A + 3B + 3C$$

$$4 = 1 + 3B - \frac{3}{2}$$

$$3B = 3 + \frac{3}{2} = \frac{9}{2}$$

$$B = \frac{3}{2}$$

$$\frac{4+12x+x^2}{(3-x)(1+2x)^2}=\frac{1}{3-x}+\frac{3}{2\,(1+2x)}-\frac{1}{2(1+2x)^2}$$

ii $\frac{1}{3-x}+\frac{3}{2\,(1+2x)}-\frac{1}{2(1+2x)^2}=(3-x)^{-1}+\frac{3}{2}(1+2x)^{-1}-\frac{1}{2}(1+2x)^{-2}$

$(3-x)^{-1}$

$=3^{-1}\left(1-\frac{x}{3}\right)^{-1}$

$=\frac{1}{3}\left\{1+(-1)\left(-\frac{x}{3}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{x}{3}\right)^2+...\right\}$

$=\frac{1}{3}+\frac{1}{9}x+\frac{1}{27}x^2+...$

$\frac{3}{2}(1+2x)^{-1}$

$=\frac{3}{2}\left\{1+(-1)(2x)+\frac{(-1)(-2)}{2!}(2x)^2+...\right\}$

$=\frac{3}{2}-3x+6x^2-...$

$\frac{1}{2}(1+2x)^{-2}$

$=\frac{1}{2}\left\{1+(-2)(2x)+\frac{(-2)(-3)}{2!}(2x)^2+...\right\}$

$=\frac{1}{2}-2x+6x^2-...$

$(3-x)^{-1}+\frac{3}{2}(1+2x)^{-1}-\frac{1}{2}(1+2x)^{-2}$

$=\left(\frac{1}{3}+\frac{1}{9}x+\frac{1}{27}x^2+...\right)+\left(\frac{3}{2}-3x+6x^2-...\right)-\left(\frac{1}{2}-2x+6x^2-...\right)$

$=\left(\frac{1}{3}+\frac{3}{2}-\frac{1}{2}\right)+\left(\frac{1}{9}-3+2\right)x+\left(\frac{1}{27}+6-6\right)x^2+...$

$=\frac{4}{3}-\frac{8}{9}x+\frac{1}{27}x^2+...$

18 **i** $f(x)=\frac{x^2-8x+9}{(1-x)(2-x)^2}=\frac{A}{1-x}+\frac{B}{2-x}+\frac{C}{(2-x)^2}$

$x^2-8x+9=A(2-x)^2+B(1-x)(2-x)+C(1-x)$

Letting $x=1$:

$2=A+0+0$

$A=2$

Letting $x=2$:

$-3=0+0-C$

$C=3$

Letting $x=0$:

$9=4A+2B+C$

$9=8+2B+3$

$2B=-2$

$B=-1$

$f(x)=\frac{x^2-8x+9}{(1-x)(2-x)^2}=\frac{2}{1-x}-\frac{1}{2-x}+\frac{3}{(2-x)^2}$

ii $\mathrm{f}(x)=\frac{x^2-8x+9}{(1-x)(2-x)^2}=\frac{2}{1-x}-\frac{1}{2-x}+\frac{3}{(2-x)^2}$

$=2(1-x)^{-1}-(2-x)^{-1}+3(2-x)^{-2}$

$2(1-x)^{-1}$

$=2\left\{1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+...\right\}$

$=2+2x+2x^2+...$

$(2-x)^{-1}$
$= 2^{-1}\left(1-\frac{x}{2}\right)^{-1}$
$= \frac{1}{2}\left\{1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\ldots\right\}$
$= \frac{1}{2}+\frac{1}{4}x+\frac{1}{8}x^2+\ldots$
$3(2-x)^{-2}$
$= 3\left(2^{-2}\right)\left(1-\frac{x}{2}\right)^{-2}$
$= \frac{3}{4}\left\{1+(-2)\left(-\frac{x}{2}\right)+\frac{(-2)(-3)}{2!}\left(-\frac{x}{2}\right)^2+\ldots\right\}$
$= \frac{3}{4}+\frac{3}{4}x+\frac{9}{16}x^2+\ldots$
$= 2(1-x)^{-1}-(2-x)^{-1}+3(2-x)^{-2}$
$= \left(2+2x+2x^2+\ldots\right)-\left(\frac{1}{2}+\frac{1}{4}x+\frac{1}{8}x^2+\ldots\right)+\left(\frac{3}{4}+\frac{3}{4}x+\frac{9}{16}x^2+\ldots\right)$
$= \left(2-\frac{1}{2}+\frac{3}{4}\right)+\left(2-\frac{1}{4}+\frac{3}{4}\right)x+\left(2-\frac{1}{8}+\frac{9}{16}\right)x^2+\ldots$
$= \frac{9}{4}+\frac{5}{2}x+\frac{39}{16}x^2+\ldots$

19 **a** When the denominator has a quadratic factor that cannot be factorised, you will need to use this form of partial fraction. When you try to find the values of A, B and C you will need to equate coefficients at some point, because there are not enough factors that can be set to zero by choice of x.

$f(x)=\frac{5x-2}{(x-1)(2x^2-1)}=\frac{A}{x-1}+\frac{Bx+C}{2x^2-1}$
$5x-2=A(2x^2-1)+(Bx+C)(x-1)$
Letting $x=1$:
$3=A+0$
$A=3$
Letting $x=0$:
$-2=-A-C$
$C=-A+2=-1$
Equating coefficients of x^2:
$0=2A+B$
$0=6+B$
$B=-6$
$f(x)=\frac{5x-2}{(x-1)(2x^2-1)}=\frac{3}{x-1}-\frac{6x+1}{2x^2-1}$

b $\mathrm{f}(x)=\frac{5x-2}{(x-1)(2x^2-1)}=\frac{3}{x-1}-\frac{6x+1}{2x^2-1}$
$= 3(x-1)^{-1}+(6x+1)\left(1-2x^2\right)^{-1}$
$3(x-1)^{-1}$
$= -3(1-x)^{-1}$
$= -3\left\{1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+\frac{(-1)(-2)(-3)}{3!}(-x)^3+\ldots\right\}$
$= -3-3x-3x^2-3x^3-\ldots$
$(6x+1)\left(1-2x^2\right)^{-1}$
$= (6x+1)\left\{1+(-1)\left(-2x^2\right)+\frac{(-1)(-2)}{2!}\left(-2x^2\right)^2+\ldots\right\}$
$= (6x+1)\left\{1+2x^2+4x^4+\ldots\right\}$
$= 1+6x+2x^2+12x^3+\ldots$

$3(x-1)^{-1}+(6x+1)(1-2x^2)^{-1}$
$=(-3-3x-3x^2-3x^3-\ldots)+(1+6x+2x^2+12x^3+\ldots)$
$=-2+3x-x^2+9x^3-\ldots$

20 **a** $\dfrac{x^3-2}{x^2(2x-1)}=\dfrac{x^3-2}{2x^3-x^2}=A+\dfrac{B}{x}+\dfrac{C}{x^2}+\dfrac{D}{2x-1}$

$x^3-2=Ax^2(2x-1)+Bx(2x-1)+C(2x-1)+Dx^2$

Letting $x=\dfrac{1}{2}$:

$\dfrac{1}{8}-2=0+0+0+\dfrac{1}{4}D$

$D=\dfrac{1}{2}-8=-\dfrac{15}{2}$

Letting $x=0$:

$-2=0+0-C+0$

$C=2$

Equating coefficients of x:

$0=-B+2C$

$B=4$

Letting $x=1$:

$-1=A+B+C+D$

$-1=A+4+2-\dfrac{15}{2}$

$A=-1-4-2+\dfrac{15}{2}=\dfrac{1}{2}$

b $\dfrac{x^3-2}{x^2(2x-1)}=\dfrac{1}{2}+\dfrac{4}{x}+\dfrac{2}{x^2}-\dfrac{15}{2\,(2x-1)}$

$$\int_1^2 \frac{x^3-2}{x^2(2x-1)}\,\mathrm{d}x=\int_1^2 \frac{1}{2}+\frac{4}{x}+\frac{2}{x^2}-\frac{15}{2\,(2x-1)}\,\mathrm{d}x$$
$$=\int_1^2\left(\frac{1}{2}+\frac{4}{x}+2x^{-2}-\frac{15}{2(2x-1)}\right)\mathrm{d}x$$

Note that all terms where the power is -1 are left as fractions, so that logarithms can be used. All other terms are rewritten using rules of indices.

$=\left[\dfrac{1}{2}x+4\ln|x|-2x^{-1}-\dfrac{15}{4}\ln|2x-1|\right]_1^2$

$=1+4\ln 2-1-\dfrac{15}{4}\ln 3-\dfrac{1}{2}-4\ln 1+2+\dfrac{15}{4}\ln 1$

$=\dfrac{3}{2}+4\ln 2-\dfrac{16}{4}\ln 31+\dfrac{1}{4}\ln 3$

$=\dfrac{3}{2}+4\ln 2-4\ln 3+\dfrac{1}{4}\ln 3$

$=\dfrac{3}{2}+2\ln\left(\dfrac{2}{3}\right)^2+\dfrac{1}{4}\ln 3$

$=\dfrac{3}{2}+2\ln\dfrac{4}{9}+\dfrac{1}{4}\ln 3$

$=\dfrac{3}{2}-2\ln\dfrac{9}{4}+\dfrac{1}{4}\ln 3$

21 **i** $y=x^2\ln x$

Using the product rule:

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=2x\ln x+x^2\times\dfrac{1}{x}$

$=2x\ln x+x$

At M, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$:

$x(2\ln x+1)=0$

$x = 0$ or $\ln x = -\frac{1}{2}$

$x > 0$ at M (from the diagram)

$x = \mathrm{e}^{-\frac{1}{2}} = \frac{1}{\sqrt{\mathrm{e}}}$

$y = \left(\mathrm{e}^{-\frac{1}{2}}\right)^2 \ln \mathrm{e}^{-\frac{1}{2}}$

$= \mathrm{e}^{-1}\left(-\frac{1}{2}\right) = -\frac{1}{2\mathrm{e}}$

The coordinates of M are $\left(\frac{1}{\sqrt{\mathrm{e}}}, -\frac{1}{2\mathrm{e}}\right)$.

ii Lower bound of region when $y = 0$.

$x^2 \ln x = 0$

$x = 0$ or $x = 1$

Shaded region is bounded by $x = 1$ and $x = \mathrm{e}$.

Using integration by parts:

$$\int_1^{\mathrm{e}} x^2 \ln x \, \mathrm{d}x$$

$$= \left[\frac{1}{3}x^3 \ln x\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} \frac{1}{3}x^3 \times \frac{1}{x} \, \mathrm{d}x$$

$$= \frac{1}{3}\mathrm{e}^3 - 0 - \int_1^{\mathrm{e}} \frac{1}{3}x^2 \, \mathrm{d}x$$

$$= \frac{1}{3}\mathrm{e}^3 - \left[\frac{1}{9}x^3\right]_1^{\mathrm{e}}$$

$$= \frac{1}{3}\mathrm{e}^3 - \frac{1}{9}\mathrm{e}^3 + \frac{1}{9}$$

$$= \frac{2}{9}\mathrm{e}^3 + \frac{1}{9}$$

$$= \frac{1}{9}\left(2\mathrm{e}^3 + 1\right)$$

22 i $\int_2^4 4x \ln x \, \mathrm{d}x$

$$= \left[2x^2 \ln x\right]_2^4 - \int_2^4 \left(2x^2 \times \frac{1}{x}\right) \mathrm{d}x$$

$$= 32 \ln 4 - 8 \ln 2 - \int_2^4 2x \, \mathrm{d}x$$

$$= 32 \ln 2^2 - 8 \ln 2 - \left[x^2\right]_2^4$$

$$= 64 \ln 2 - 8 \ln 2 - (16 - 4)$$

$$= 56 \ln 2 - 12$$

ii $u = \sin 4x$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 4 \cos 4x$$

$$\frac{1}{4} \, \mathrm{d}u = \cos 4x \, \mathrm{d}x$$

$$1 - u^2 = 1 - \sin^2 4x$$

$$= \cos^2 4x$$

$$\frac{1}{4}\left(1 - u^2\right) \mathrm{d}u = \cos^3 4x \, \mathrm{d}x$$

$x = 0 \Rightarrow u = 0$

$x = \frac{\pi}{24} \Rightarrow u = \sin \frac{\pi}{6} = \frac{1}{2}$

$$\int_0^{\frac{\pi}{24}} \cos^3 4x \, dx = \int_0^{\frac{1}{2}} \frac{1}{4}(1-u^2) \, du$$

$$= \left[\frac{1}{4}u - \frac{1}{12}u^3\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{8} - \frac{1}{96}$$

$$= \frac{11}{96}$$

23 i $4\cos\theta + 3\sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$R\sin\alpha = 3$

$R\cos\alpha = 4$

$R^2 = 3^2 + 4^2 = 25$

$R = 5$

$\tan\alpha = \frac{3}{4}$

$\alpha = 0.6435$

$4\cos\theta + 3\sin\theta = 5\cos(\theta - 0.6435)$

ii a Using the result from part **i**:

$5\cos(\theta - 0.6435) = 2$

$\cos(\theta - 0.6435) = \frac{2}{5}$

$\theta - 0.6435 = 1.159279...$ or $\theta - 0.6435 = 5.1239058...$

$\theta = 1.80$ $\theta = 5.77$

b Using the result from part **i**:

$$\int \frac{50}{(4\cos\theta + 3\sin\theta)^2} \, d\theta$$

$$= \int \frac{50}{25\cos^2(\theta - 0.6435)} d\theta$$

$$= \int 2\sec^2(\theta - 0.6435) \, d\theta$$

$$= 2\tan(\theta - 0.6435) + C$$

Chapter 9
Vectors

P3 **This chapter is for Pure Mathematics 3 students only.**

EXERCISE 9A

1 **a** From A and B you move right by 5 units and down by 3 units.

$\overrightarrow{AB} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

From B to C you move right by 5 units and up by 2 units.

$\overrightarrow{BC} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

b $\overrightarrow{AC} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \end{pmatrix}$

2 **a** $\overrightarrow{EF} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$ because to get from E to F you move left by 7 units and up by 3.

b $\overrightarrow{DF} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$\overrightarrow{DE} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$

$\overrightarrow{DF} - \overrightarrow{DE} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 10 \\ 2 \end{pmatrix} = \begin{pmatrix} -7 \\ 3 \end{pmatrix} = \overrightarrow{EF}$ from part **a**.

3 You get from Q to R by first moving from Q to P and then from P to R.

> Try to show the route that you have taken as clearly as possible.

So $\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PR}$. But then $\overrightarrow{QP} = -\overrightarrow{PQ}$ so

$\overrightarrow{QR} = -\overrightarrow{PQ} + \overrightarrow{PR} = \overrightarrow{PR} - \overrightarrow{PQ}$

4 **a**

$$\begin{aligned} \overrightarrow{XY} &= \overrightarrow{XA} + \overrightarrow{AY} \\ &= -\mathbf{a} + \mathbf{b} \\ \overrightarrow{BC} &= \overrightarrow{BA} + \overrightarrow{AC} \\ &= -2\mathbf{a} + 2\mathbf{b} \\ &= 2(-\mathbf{a} + \mathbf{b}) \\ &= 2\overrightarrow{XY} \end{aligned}$$

$\overrightarrow{BC}$ is a multiple of $\overrightarrow{XY}$, so the vectors are parallel.

> Remember that two vectors are multiples of one another if, and only if, they are parallel.

b Using the result from part **a**:

$$\begin{aligned} \overrightarrow{BC} &= 2\overrightarrow{XY} \\ \overrightarrow{XY} &= \frac{1}{2}\overrightarrow{BC} \\ |\overrightarrow{XY}| &= \frac{1}{2}|\overrightarrow{BC}| \\ k &= \frac{1}{2} \end{aligned}$$

5 **a** Note that $\overrightarrow{AB}$ is the component of $\overrightarrow{AG}$ parallel to the x-axis, $\overrightarrow{BC}$ is the component of $\overrightarrow{AG}$ parallel to the y-axis and $\overrightarrow{CG}$ is the component of $\overrightarrow{AG}$ parallel to the z-axis. $\overrightarrow{AG}$ is given, so you can work out vectors representing each side of the cuboid.

$$\begin{aligned}\overrightarrow{AM} &= \overrightarrow{AB} + \overrightarrow{BM}\\ &= \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}\\ &= \begin{pmatrix}12\\0\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\4\\0\end{pmatrix}\\ &= \begin{pmatrix}12\\2\\0\end{pmatrix}\end{aligned}$$

b
$$\begin{aligned}\overrightarrow{AN} &= \overrightarrow{AD} + \overrightarrow{DF} + \overrightarrow{FN}\\ &= \overrightarrow{AD} + \overrightarrow{DF} + \frac{1}{4}\overrightarrow{FG}\\ &= \begin{pmatrix}0\\4\\0\end{pmatrix} + \begin{pmatrix}0\\0\\2\end{pmatrix} + \frac{1}{4}\begin{pmatrix}12\\0\\0\end{pmatrix}\\ &= \begin{pmatrix}3\\4\\2\end{pmatrix}\end{aligned}$$

6 **a** **i**
$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BA} + \overrightarrow{AH} + \overrightarrow{HC}\\ &= -\mathbf{p} + \mathbf{q} + \mathbf{s}\end{aligned}$$

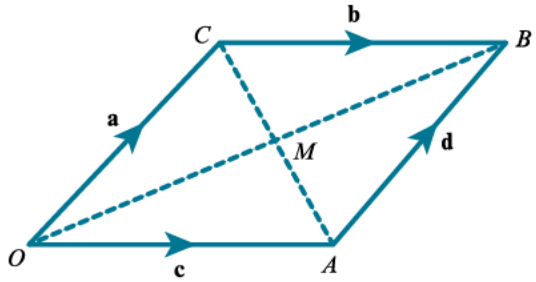
ii Note that $\overrightarrow{FG}$ is the same as $\overrightarrow{CB}$. But $\overrightarrow{CB}$ is $-\overrightarrow{BC}$ because the direction is reversed. You have already calculated $\overrightarrow{BC}$ in part **i**.

$$\begin{aligned}\overrightarrow{EB} &= \overrightarrow{EF} + \overrightarrow{FG} + \overrightarrow{GH} + \overrightarrow{HA} + \overrightarrow{AB}\\ &= -\mathbf{p} - (-\mathbf{p} + \mathbf{q} + \mathbf{s}) - \mathbf{r} - \mathbf{q} + \mathbf{p}\\ &= -\mathbf{p} + \mathbf{p} - \mathbf{q} - \mathbf{s} - \mathbf{r} - \mathbf{q} + \mathbf{p}\\ &= \mathbf{p} - \mathbf{r} - \mathbf{s} - 2\mathbf{q}\end{aligned}$$

b For example, angle $AHC = 45^\circ$ (interior angle of a regular octagon $= 135^\circ$ and angle $GHC = 90^\circ$) and the exterior angle (at A) is 45° and so the line segments AB and HC are parallel.

$$\begin{aligned}|\mathbf{s}| &= |\mathbf{p}| + 2\,|\mathbf{q}|\ \cos 45^\circ\\ |\mathbf{p}| &= |\mathbf{q}|\\ |\mathbf{s}| &= |\mathbf{p}| + |\mathbf{p}|\,\sqrt{2} = (1+\sqrt{2})\,|\mathbf{p}|\\ \mathbf{s} &= (1+\sqrt{2})\mathbf{p}\\ k &= 1+\sqrt{2}\end{aligned}$$

7



$$\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

$$\overrightarrow{AM} = \overrightarrow{MC}$$
$$\overrightarrow{AO} + \overrightarrow{OM} = \overrightarrow{MO} + \overrightarrow{OC}$$
$$-\mathbf{c} + \frac{1}{2}(\mathbf{a}+\mathbf{b}) = -\frac{1}{2}(\mathbf{a}+\mathbf{b}) + \mathbf{a}$$
$$\mathbf{a}+\mathbf{b} = \mathbf{a}+\mathbf{c}$$
$$\mathbf{b} = \mathbf{c}$$

The sides OA and BC are parallel.

$\overrightarrow{OM} = \overrightarrow{MB}$

$\frac{1}{2}(\mathbf{a}+\mathbf{b}) = \frac{1}{2}(\mathbf{c}+\mathbf{d})$

$\mathbf{a}+\mathbf{b} = \mathbf{c}+\mathbf{d}$

But

$$\mathbf{b} = \mathbf{c}$$
$$\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{d}$$
$$\mathbf{a} = \mathbf{d}$$

The sides OC and AB are parallel.

EXERCISE 9B

1 a

$$\begin{aligned}\overrightarrow{AB} &= \mathbf{b} - \mathbf{a}\\ &= (5\mathbf{i} + 7\mathbf{k}) - (5\mathbf{i} + 3\mathbf{j})\\ &= -3\mathbf{j} + 7\mathbf{k}\\ |\overrightarrow{AB}| &= \sqrt{(-3)^2 + 7^2} = \sqrt{58}\\ \text{Unit vector } &= \frac{1}{\sqrt{58}}(-3\mathbf{j} + 7\mathbf{k})\end{aligned}$$

Given that you need to underline vectors when you write them for yourself, it is best to write each term as a fraction followed by the vector rather than a vector divided by the magnitude. For example, $\frac{1}{2}\mathbf{a}$ rather than $\frac{\mathbf{a}}{2}$.

b

$$\begin{aligned}|\overrightarrow{OA}| &= \sqrt{5^2 + 3^2} = \sqrt{34}\\ |\overrightarrow{AB}| &= |5\mathbf{i} + \lambda\mathbf{j} + 10\mathbf{k} - (5\mathbf{i} + 3\mathbf{j})|\\ &= \sqrt{(\lambda - 3)^2 + 10^2}\\ &= \sqrt{(\lambda - 3)^2 + 100}\\ Area &= \frac{1}{2}|\overrightarrow{OA}|\,|\overrightarrow{AB}|\\ &= \frac{1}{2}\sqrt{34}\sqrt{(\lambda - 3)^2 + 100} = 5\sqrt{34}\end{aligned}$$

$$\sqrt{(\lambda - 3)^2 + 100} = 10$$

$$(\lambda - 3)^2 + 100 = 100$$

$$\lambda = 3$$

2 a Height of cuboid $=$ $\mathbf{k}$ component of $\overrightarrow{CE}$

$= 2 \times 1 = 2$

b

$$\begin{aligned}\overrightarrow{ON} &= \overrightarrow{OC} + \overrightarrow{CG} + \overrightarrow{GN}\\ &= 4\mathbf{j} + 2\mathbf{k} + 6\mathbf{i}\\ &= 6\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}\end{aligned}$$

c

$$\begin{aligned}\overrightarrow{MN} &= \overrightarrow{MO} + \overrightarrow{ON}\\ &= -(4\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + 6\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}\\ &= 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}\\ |\overrightarrow{MN}| &= \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3\\ \text{Unit vector } &= \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})\end{aligned}$$

3 a i

$$|\overrightarrow{AD}| = |\mathbf{d} - \mathbf{a}| = \left|\begin{pmatrix}-6\\-6\\0\end{pmatrix} - \begin{pmatrix}2\\2\\0\end{pmatrix}\right| = \left|\begin{pmatrix}-8\\-8\\0\end{pmatrix}\right| = \sqrt{128} = 8\sqrt{2}$$

$$|\overrightarrow{BC}| = |\mathbf{c} - \mathbf{b}| = \left|\begin{pmatrix}5\\-3\\4\end{pmatrix} - \begin{pmatrix}13\\5\\4\end{pmatrix}\right| = \left|\begin{pmatrix}-8\\-8\\0\end{pmatrix}\right| = 8\sqrt{2}$$

$$|\overrightarrow{AB}| = |\mathbf{b} - \mathbf{a}| = \left|\begin{pmatrix}13\\5\\4\end{pmatrix} - \begin{pmatrix}2\\2\\0\end{pmatrix}\right| = \left|\begin{pmatrix}11\\3\\4\end{pmatrix}\right| = \sqrt{11^2 + 3^2 + 4^2} = \sqrt{146}$$

$$|\overrightarrow{DC}| = |\mathbf{c} - \mathbf{d}| = \left|\begin{pmatrix}5\\-3\\4\end{pmatrix} - \begin{pmatrix}-6\\-6\\0\end{pmatrix}\right| = \left|\begin{pmatrix}11\\3\\4\end{pmatrix}\right| = \sqrt{11^2 + 3^2 + 4^2} = \sqrt{146}$$

ii From the results in part **i**, opposite sides are parallel and equal in length, so $ABCD$ is a parallelogram.

b **i** $\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 11 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7.5 \\ 3.5 \\ 2 \end{pmatrix}$

so $M\,(7.5,3.5,2)$

ii $\overrightarrow{OP} = \overrightarrow{OB} + \frac{1}{3}\overrightarrow{BD}$

$$= \begin{pmatrix} 13 \\ 5 \\ 4 \end{pmatrix} + \frac{1}{3}\left[\begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix} - \begin{pmatrix} 13 \\ 5 \\ 4 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 13 \\ 5 \\ 4 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -19 \\ -11 \\ -4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 20 \\ 4 \\ 8 \end{pmatrix}$$

so $P\left(\frac{20}{3}, \frac{4}{3}, \frac{8}{3}\right)$

4 **a** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) - (\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}) = 2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$

$\left|\overrightarrow{AB}\right| = |2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}| = \sqrt{4 + 36 + 16} = \sqrt{56} = 2\sqrt{14}$

b $\left|\overrightarrow{OA}\right| = \sqrt{1 + 4 + 25} = \sqrt{30}$

$\left|\overrightarrow{OB}\right| = \sqrt{9 + 16 + 1} = \sqrt{26}$

$30 + 26 = 56$

So by Pythagoras' theorem, OAB is a right-angled triangle.

c Area $= \frac{1}{2} \times \sqrt{30} \times \sqrt{26} = \sqrt{195}$

5 $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (7\mathbf{i} + 4\mathbf{j} - \mathbf{k}) - (5\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

$= 2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = (\mathbf{i} + \mathbf{j} + q\mathbf{k}) - (5\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

$= -4\mathbf{i} + (q - 2)\mathbf{k}$

$\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{AC}\right|^2$

$2^2 + 3^2 + (-3)^2 = (-4)^2 + (q - 2)^2$

$16 + (q - 2)^2 = 22$

$(q - 2)^2 = 6$

$q = 2 \pm \sqrt{6}$

If two magnitudes are equal then so are their squares. This saves you having to use square roots!

6 **a** $\overrightarrow{DB} = -7\mathbf{k} + 24\mathbf{i}$

$\left|\overrightarrow{DB}\right| = \sqrt{(-7)^2 + 24^2} = 25$ cm

b $\overrightarrow{ON} = \overrightarrow{OD} + \frac{2}{5}\overrightarrow{DB}$

$= 20\mathbf{j} + 7\mathbf{k} + \frac{2}{5}(-7\mathbf{k} + 24\mathbf{i})$

$= 9.6\mathbf{i} + 20\mathbf{j} + 4.2\mathbf{k}$

7 $\left|\overrightarrow{OP}\right| = \left|\overrightarrow{OQ}\right|$

$\sqrt{4 + 25 + a^2} = \sqrt{1 + (1 + a)^2 + (-3)^2}$

$4 + 25 + a^2 = 1 + (1 + a)^2 + (-3)^2$

$a^2 + 29 = a^2 + 2a + 1 + 1 + 9$

$2a = 18$

$a = 9$

8 **a** $\overrightarrow{OP} = \lambda\overrightarrow{OQ}$ and using the y-component, $\lambda = \frac{1}{4}$.

Hence, $-6k = \frac{1}{4}(2k+13)$, $k = -\frac{1}{2}$

and checking $8\,(1+k) = \frac{1}{4}(-32k)$ gives $k = -\frac{1}{2}$.

b $\overrightarrow{OP} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

and

$\overrightarrow{OQ} = \begin{pmatrix} 12 \\ -8 \\ 16 \end{pmatrix} = 12\mathbf{i} - 8\mathbf{j} + 16\mathbf{k}$

c $\overrightarrow{PQ} = 9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}$ and

$\left|\overrightarrow{PQ}\right| = \sqrt{9^2 + (-6)^2 + 12^2} = 3\sqrt{29}$

9 Home is the null displacement $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Total vector sum is $\begin{pmatrix} 13 \\ 8 \\ 0 \end{pmatrix}$ so to get home the displacement vector is $\begin{pmatrix} -13 \\ -8 \\ 0 \end{pmatrix}$.

The distance home is $10\sqrt{(-13)^2 + (-8)^2 + 0^2} = 10\sqrt{169+64} = 153$ cm to the nearest cm.

EXERCISE 9C

1 Remember that two vectors are perpendicular if, and only if, their scalar product is zero.

a
$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= 2\times 7+8\times(-1)+-2\times 3\\ &= 14-8-6=0\end{aligned}$$

Perpendicular

b
$$\begin{aligned}\mathbf{c}\cdot\mathbf{d} &= 5\times(-3)+12\times(-1)+13\times 3\\ &= -15-12+39=12\\ 12 &= \sqrt{5^2+12^2+13^3}\sqrt{(-3)^2+(-1)^2+3^2}\ \cos\theta\\ 12 &= \left(13\sqrt{2}\right)\sqrt{19}\ \cos\theta\\ \cos\theta &= \frac{12}{13\sqrt{38}}\\ \theta &= 81.4^\circ(1\text{ d.p.})\end{aligned}$$

Check the question for any indication that you need an acute angle. If your calculations produce an obtuse result, you should subtract it from 180 to get the required acute angle.

c
$$\begin{aligned}\mathbf{e}\cdot\mathbf{f} &= 4\times(-4)+-9\times(-2)+-2\times 1\\ &= -16+18-2=0\end{aligned}$$

Perpendicular

2
$$\begin{aligned}\overrightarrow{OB}\cdot\overrightarrow{OA} &= -5\times 1+0\times 7+3\times 2\\ &= \sqrt{(-5)^2+0^2+3^2}\sqrt{1^2+7^2+2^2}\ \cos BOA\\ 1 &= \sqrt{34}\sqrt{54}\ \cos BOA\\ \cos BOA &= \frac{1}{\sqrt{34}\sqrt{54}}\\ BOA &= 88.7^\circ(1\text{ to d.p.})\end{aligned}$$

3
$$\begin{aligned}6a+(-2)(4)+(5)(-2) &= 0\\ a &= 3\end{aligned}$$

4 **a**
$$\begin{aligned}5k^2-3(k+2)-(7k+9) &= 0\\ 5k^2-10k-15 &= 0\\ k^2-2k-3 &= 0\\ (k+1)(k-3) &= 0\end{aligned}$$

$k=-1$ or $k=3$

b
$$\overrightarrow{OP}=\begin{pmatrix}10\\-3\\23\end{pmatrix},\overrightarrow{OQ}=\begin{pmatrix}2\\4\\-1\end{pmatrix}$$

$$\begin{aligned}\overrightarrow{OP}\cdot\overrightarrow{OQ} &= 10(2)+(-3)(4)+(23)(-1)=-15\\ \left|\overrightarrow{OP}\right| &= \sqrt{10^2+(-3)^2+23^2}=\sqrt{638}\\ \left|\overrightarrow{OQ}\right| &= \sqrt{2^2+4^2+(-1)^2}=\sqrt{21}\end{aligned}$$

$$\theta=\cos^{-1}\left(\frac{-15}{\sqrt{638}\sqrt{21}}\right)=97.4^\circ\ \text{(to 1 decimal place)}$$

5 $\overrightarrow{NP}=2\mathbf{j}+3\mathbf{k}$ and $\overrightarrow{MP}=-3\mathbf{i}-2\mathbf{j}+\mathbf{k}$

and so $\overrightarrow{NP}\cdot\overrightarrow{MP}=2(-2)+3(1)=-1$

Also $\left|\overrightarrow{NP}\right|=\sqrt{13}$ and $\left|\overrightarrow{MP}\right|=\sqrt{14}$

So $NPM=\cos^{-1}\left(\frac{-1}{\sqrt{13}\sqrt{14}}\right)=94.3^\circ$ (to 1 decimal place)

6 $\mathbf{a} \cdot \mathbf{j} = (4)(0) + (-8)(1) + (1)(0) = -8$

$|\mathbf{a}| = \sqrt{4^2 + (-8)^2 + 1^2} = \sqrt{81} = 9$

$|\mathbf{j}| = 1$

$\theta = \cos^{-1}\left(\frac{-8}{9}\right) = 152.7^\circ$ (to 1 decimal place)

7 $\mathbf{a} \cdot \mathbf{b}$ is a scalar and the dot product is a product of two vectors.

8 a $\overrightarrow{OM} = \overrightarrow{OD} + \overrightarrow{DG} + \overrightarrow{GM}$

$= \overrightarrow{OD} + \overrightarrow{DG} + \frac{1}{2}\overrightarrow{GF}$

$= 4\mathbf{k} + 4\mathbf{j} + 2\mathbf{i}$

$= 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$

$\overrightarrow{NG} = \overrightarrow{NB} + \overrightarrow{BC} + \overrightarrow{CG}$

$= 3\mathbf{j} - 4\mathbf{i} + 4\mathbf{k}$

$= -4\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

b $\overrightarrow{OM} \cdot \overrightarrow{NG} = 2(-4) + 4(3) + 4(4) = 20$

$\left|\overrightarrow{OM}\right| = 6$

$\left|\overrightarrow{NG}\right| = \sqrt{41}$

$\cos^{-1}\left(\frac{20}{6\sqrt{41}}\right) = 58.6^\circ$ (to 1 decimal place)

9 $\overrightarrow{AM} = -77\mathbf{i} + 30\mathbf{j} + 36\mathbf{k}$ and

$\left|\overrightarrow{DB}\right| = \sqrt{77^2 + 36^2} = 85$

so $\overrightarrow{AN} = 60\mathbf{j} + \frac{1}{5}\overrightarrow{BD} = 60\mathbf{j} + \frac{1}{5}(-77\mathbf{i} + 36\mathbf{k})$

$= -\frac{77}{5}\mathbf{i} + 60\mathbf{j} + \frac{36}{5}\mathbf{k}$

$\overrightarrow{AM} \cdot \overrightarrow{AN} = -77\left(-\frac{77}{5}\right) + 30(60) + 36\left(\frac{36}{5}\right) = 3245$

$\left|\overrightarrow{AM}\right| = 25\sqrt{13}$

$\left|\overrightarrow{AN}\right| = \sqrt{3889}$

$MAN = \cos^{-1}\left(\frac{3245}{25\sqrt{13}\sqrt{3889}}\right) = 54.7^\circ$ (to 1 decimal place)

10 a $\overrightarrow{AN} = \begin{pmatrix} -3 \\ 1.5 \\ 4.5 \end{pmatrix}, \left|\overrightarrow{AN}\right| = \frac{3\sqrt{14}}{2}$

$\cos^{-1}\left(\frac{4.5}{3 \times 1.5\sqrt{14}}\right) = 74.5^\circ$ (to 1 decimal place)

b $\overrightarrow{MN} = \begin{pmatrix} -3 \\ 0 \\ 4.5 \end{pmatrix}$

c $\overrightarrow{PN} \cdot \overrightarrow{MN} = 0$

$\overrightarrow{PN} = \begin{pmatrix} 3 \\ 1.5 \\ 4.5 - p \end{pmatrix}$

$3(-3) + 4.5(4.5 - p) = 0$

$p = 2.5$

$\overrightarrow{OP} = \begin{pmatrix} 0 \\ 0 \\ 2.5 \end{pmatrix}$

EXERCISE 9D

1

Remember that the vector equation takes the form

$\mathbf{r} =$ position vector of known point $+ t$(direction vector).

a $\mathbf{r} = -\mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + 6\mathbf{j} - \mathbf{k})$

b $\mathbf{r} = \lambda(7\mathbf{i} - \mathbf{j} - \mathbf{k})$

c $\mathbf{r} = 7\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \lambda(3\mathbf{i} - 4\mathbf{k})$

2

Remember that $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, so you simply need to equate each of x, y, z with the number of $\mathbf{i}s, \mathbf{j}s, \mathbf{k}s$ respectively.

a $x = 2\lambda$
$y = -1 + 6\lambda$
$z = 5 - \lambda$

b $x = 7\lambda$
$y = -\lambda$
$z = -\lambda$

c $x = 7 + 3\lambda$
$y = 2$
$z = -3 - 4\lambda$

3 $9\mathbf{i} + 2\mathbf{j} - 5\mathbf{k} - (\mathbf{i} + 7\mathbf{j} + \mathbf{k})$
$= 8\mathbf{i} - 5\mathbf{j} - 6\mathbf{k}$
$= \frac{1}{2}(16\mathbf{i} - 10\mathbf{j} - 12\mathbf{k})$
So the direction vectors of the two lines, and hence the two lines themselves, are parallel.

4 a $x = 2 + t$
$y = 13 + t$
$z = 1 - t$

b $x = 2t$
$y = 10 + 5t$
$z = 0$

c $x = 1 + 2t$
$y = -3 + 3t$
$z = 4t$

5 a $\overrightarrow{AB} = \begin{pmatrix} 1-0 \\ 1-4 \\ 6-(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 8 \end{pmatrix}$

So $\overrightarrow{OA} + t\overrightarrow{AB} = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 8 \end{pmatrix} = \begin{pmatrix} t \\ 4-3t \\ -2+8t \end{pmatrix}$

and so

$x = t$
$y = 4 - 3t$
$z = -2 + 8t$

b At the point where L crosses the Oxy plane $z = 0$

So $-2 + 8t = 0$

So $t = \frac{1}{4}$

So $x = \frac{1}{4}$, $y = 4 - \frac{3}{4} = \frac{13}{4}$

The point has coordinates $\left(\frac{1}{4}, \frac{13}{4}, 0\right)$.

6 **a** $\mathbf{r} = (\mu + 4)\,\mathbf{i} + (\mu - 7)\,\mathbf{j} + (3\mu)\,\mathbf{k}$

$\mathbf{r} = 4\mathbf{i} - 7\mathbf{j} + \mu\,(\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

The direction is $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, which is not a scalar multiple of $6\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, so the lines are not parallel.

b Direction vectors are:

$\mathbf{a} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

$$\mathbf{a} \cdot \mathbf{b} = 1 \times 6 + 1 \times 1 + 3 \times 3 = \sqrt{1+1+9}\sqrt{36+1+9}\ \cos\theta$$
$$16 = \sqrt{11}\sqrt{46}\ \cos\theta$$

$\theta = \cos^{-1}\left(\frac{16}{\sqrt{11}\sqrt{46}}\right) = 44.7^\circ$ (to 1 decimal place)

7 **a** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$

b $\overrightarrow{BA} \cdot \mathbf{d}_{L_2} = \begin{pmatrix} 5-4 \\ -3-(-7) \\ 2-2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} = 1\,(4) + 4\,(-1) + 0\,(-3) = 0$✓

8 **a** $\overrightarrow{AB} = \begin{pmatrix} 0-3 \\ -1-1 \\ 2-5 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -3 \end{pmatrix}$

So $\overrightarrow{OB} + t\overrightarrow{AB} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -3 \\ -2 \\ -3 \end{pmatrix}$

b $\overrightarrow{ON} = \begin{pmatrix} -3t+3 \\ -2t+1 \\ -3t+5 \end{pmatrix}$

and the $\overrightarrow{CN} = \begin{pmatrix} -3t+2 \\ -2t-1 \\ -3t+2 \end{pmatrix}$

Since $\overrightarrow{CN}$ is perpendicular to L:

$$(-3t+2)\,(-3) + (-2t-1)\,(-2) + (-3t+2)\,(-3) = 0$$
$$22t - 10 = 0$$

so $t = \frac{5}{11}$

$$\overrightarrow{ON} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + \frac{5}{11}\begin{pmatrix} -3 \\ -2 \\ -3 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 18 \\ 1 \\ 30 \end{pmatrix}$$

EXERCISE 9E

1

There are two possible situations in which lines do not meet: they could be parallel or they could be skew.

a Equating components for x:

$3\lambda + 4 = \mu + 1 \Rightarrow 3\lambda - \mu = -3$ .... [1]

Equating components for y:

$5\lambda + 1 = \mu + 4 \Rightarrow 5\lambda - \mu = 3$ .... [2]

[2] – [1]:

$2\lambda = 6$

$\lambda = 3$

$\mu = 9 + 3$ from [1]

$\mu = 12$

Looking at the components of z:

$4\lambda - 3 = 9$

$2\mu + 5 = 29 \neq 9$

Equations are inconsistent, so the lines do not meet.

Direction vectors are $3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$ and $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, which are not parallel.

The lines are skew.

b Direction vectors are $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 9\mathbf{k}$ and $\mathbf{b} = -6\mathbf{i} + 8\mathbf{j} - 18\mathbf{k}$

$\mathbf{b} = -2\mathbf{a}$

The lines are parallel.

The lines also have the point with position vector $\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$ in common.

They are both equations of the same line.

2 $2 + \lambda = 11 - \mu \Rightarrow \lambda + \mu = 9$ .... [1]

$9 - 4\lambda = 9 - 2\mu \Rightarrow 2\lambda = \mu$ .... [2]

Substituting [2] into [1]:

$3\lambda = 9$

$\lambda = 3$

$\mu = 6$

$1 + 5\lambda = p + 16\mu$

$1 + 15 = p + 96$

$p = -80$

Intersect at the point with position vector

$2\mathbf{i} + 9\mathbf{j} + \mathbf{k} + 3(\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$

$= 5\mathbf{i} - 3\mathbf{j} + 16\mathbf{k}$

If you are told that the lines meet, you do not need to check your solutions in the third equation.

3 **a** First pair:

$16 - 12\lambda = 16 + 8\mu$

$-3\lambda = 2\mu$ .... [1]

$-4 + 4\lambda = 28 + 8\mu$

$4\lambda - 8\mu = 32$

$\lambda - 2\mu = 8$ .... [2]

Substituting [1] into [2]:

$$4\lambda = 8$$
$$\lambda = 2$$
$$\mu = -\frac{6}{2}$$

Point of intersection has position vector $16\mathbf{i} - 4\mathbf{j} - 6\mathbf{k} + 2(-12\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$

$= -8\mathbf{i} + 4\mathbf{j}$

You can repeat this working for the first and third lines to get the point with position vector $4\mathbf{i} - 3\mathbf{k}$. The values of the parameters would be $\lambda = 1, \mu = \frac{3}{4}$. For the second and third lines, the intersection happens when $\mu = -2, \mu = -\frac{1}{4}$, giving the point with position vector $12\mathbf{j} + 5\mathbf{k}$. Try this for yourself.

b The interior angles are the acute angles between lines.

First pair:

$$(-12\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) . (8\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}) = \sqrt{144 + 16 + 9}\sqrt{64 + 64 + 25}\ \cos\theta$$
$$\cos\theta = \frac{-96 + 32 + 15}{\sqrt{169}\sqrt{153}}$$
$$\theta = 107.7^\circ$$

Acute angle $= 180^\circ - \theta = 72.3^\circ$ (to 1 decimal place)

Similarly for the other two pairs, giving the angles 55.8° for the first and third lines and 51.9° for the second and third lines.

c The vector representing any given side is the difference between the position vectors of the endpoints.

$4\mathbf{i} - 3\mathbf{k} - (-8\mathbf{i} + 4\mathbf{j})$

$= 12\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$

$$\text{Length} = \sqrt{12^2 + (-4)^2 + (-3)^2}$$
$$= \sqrt{144 + 16 + 9} = 13$$

$4\mathbf{i} - 3\mathbf{k} - (12\mathbf{j} + 5\mathbf{k})$

$= 4\mathbf{i} - 12\mathbf{j} - 8\mathbf{k}$

$$\text{Length} = \sqrt{4^2 + (-12)^2 + (-8)^2}$$
$$= \sqrt{16 + 144 + 64} = \sqrt{224} = 4\sqrt{14}$$

$-8\mathbf{i} + 4\mathbf{j} - (12\mathbf{j} + 5\mathbf{k})$

$= -8\mathbf{i} - 8\mathbf{j} - 5\mathbf{k}$

$$\text{Length} = \sqrt{(-8)^2 + (-8)^2 + (-5)^2}$$
$$= \sqrt{64 + 64 + 25} = \sqrt{153} = 3\sqrt{17}$$

4 **a** Direction vector $= 3\mathbf{i} + 7\mathbf{j} + 9\mathbf{k} - (-\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$

$$= 4\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$$
$$\mathbf{r} = 3\mathbf{i} + 7\mathbf{j} + 9\mathbf{k} + \lambda(4\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$$

b

$$1 = 3 + 4\lambda$$
$$\lambda = -\frac{1}{2}$$
$$2 + 3\mu = 7 + 4\lambda$$
$$2 + 3\mu = 5$$
$$\mu = 1$$
$$1 + 2\mu = 3$$
$$9 + 5\lambda = \frac{13}{2} \neq 3$$

Inconsistent equations. The lines do not intersect.

5 **a** $\overrightarrow{AB} = -2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

b $\mathbf{r} = -\mathbf{i} + 2\mathbf{j} + 9\mathbf{k} + \lambda(-2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$

c $(-2\mathbf{i}+2\mathbf{j}+4\mathbf{k}).(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})=\sqrt{4+4+16}\sqrt{1+4+9}\ \cos\theta$

$$\cos\theta=\frac{2-4+12}{\sqrt{24}\sqrt{14}}$$

$$\theta=56.9^\circ\ \text{(to 1 decimal place)}$$

d $-\mathbf{i}+2\mathbf{j}+9\mathbf{k}+\lambda(-2\mathbf{i}+2\mathbf{j}+4\mathbf{k})=\mathbf{i}+3\mathbf{j}+4\mathbf{k}+\mu(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})$

$-1-2\lambda=1-\mu$

$2\lambda-\mu=-2$ .... [1]

$2+2\lambda=3-2\mu$

$2\lambda+2\mu=1$ .... [2]

[2] − [1]:

$3\mu=3$

$\mu=1$

Point of intersection has position vector

$\mathbf{i}+3\mathbf{j}+4\mathbf{k}+(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})$

$=\mathbf{j}+7\mathbf{k}$

END-OF-CHAPTER REVIEW EXERCISE 9

P3 This exercise is for Pure Mathematics 3 students only.

1 **a** $\overrightarrow{AB} = \begin{pmatrix} 3m \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 3m-2 \\ -2 \\ -6 \end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{OC}$:

$\begin{pmatrix} 3m-2 \\ -2 \\ -6 \end{pmatrix} = \begin{pmatrix} 4 \\ -m \\ -m(m+1) \end{pmatrix}$

$m = 2$

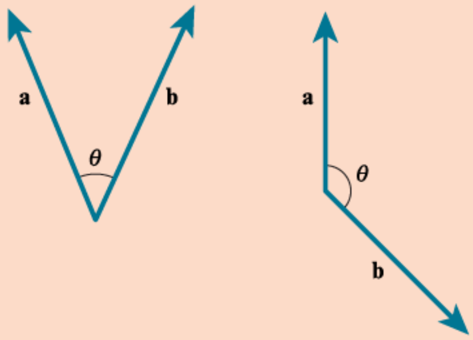
$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -2 \\ -6 \end{pmatrix} \overrightarrow{AO} = \begin{pmatrix} -2 \\ -3 \\ -7 \end{pmatrix}$

$\begin{pmatrix} 4 \\ -2 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -3 \\ -7 \end{pmatrix} = \sqrt{16+4+36}\sqrt{4+9+49}\cos OAB$

$-8+6+42 = \sqrt{56}\sqrt{62}\cos OAB$

$OAB = \cos^{-1}\left(\frac{40}{\sqrt{56}\sqrt{62}}\right) = 47.2^\circ$ (to 1 decimal place)

Remember that, when you take the scalar product of two vectors, the angle used in the definition $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$ is the angle produced when the two vectors are drawn tail to tail:



b $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$

$= \begin{pmatrix} 4 \\ -2 \\ -6 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -13 \end{pmatrix}$

Equation of line:

$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -5 \\ -13 \end{pmatrix}$

Remember that you can use any positive or negative multiple of the direction vector. It will still be an equation of the same line.

2 **a** $\overrightarrow{OA}\cdot\overrightarrow{OB} = (-2)(1) + (0)(-1) + (6)(4) = 22 \neq 0$

Vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are not perpendicular.

b **i** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$

ii $\mathbf{r} = -2\mathbf{i} + 6\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

c You can switch to column vectors if you find them easier to work with, as shown here.

$$\begin{pmatrix}-2\\0\\6\end{pmatrix} + s\begin{pmatrix}3\\-1\\-2\end{pmatrix} = \begin{pmatrix}7\\-12\\7\end{pmatrix} + t\begin{pmatrix}-3\\10\\-5\end{pmatrix}$$

$$-2 + 3s = 7 - 3t$$

$$3s + 3t = 9$$

$s + t = 3$ .... [1]

$-s = -12 + 10t$

$s + 10t = 12$ .... [2]

$6 - 2s = 7 - 5t$

$2s - 5t = -1$ .... [3]

[2] − [1]:

$9t = 9$

$t = 1$

Then from [1]:

$s = 2$

Checking [3]:

$2s - 5t = 4 - 5 = -1$

This is consistent, so the lines intersect.

Position vector of the point of intersection is:

$$\begin{pmatrix}-2\\0\\6\end{pmatrix} + s\begin{pmatrix}3\\-1\\-2\end{pmatrix}$$

$$= \begin{pmatrix}-2\\0\\6\end{pmatrix} + 2\begin{pmatrix}3\\-1\\-2\end{pmatrix}$$

$$= \begin{pmatrix}4\\-2\\2\end{pmatrix}$$

Point has coordinates $(4, -2, 2)$.

3 a $\overrightarrow{AH} = \overrightarrow{AO} + \overrightarrow{OE} + \overrightarrow{EH}$

$= -9\mathbf{i} + 12\mathbf{k} + 15\mathbf{j}$

$= -9\mathbf{i} + 15\mathbf{j} + 12\mathbf{k}$

$\overrightarrow{NH} = \overrightarrow{NE} + \overrightarrow{EH}$

$= \frac{1}{2}\overrightarrow{DE} + \overrightarrow{EH}$

$= \frac{1}{2}(5\mathbf{i} + 12\mathbf{k}) + 15\mathbf{j}$

$= 2.5\mathbf{i} + 15\mathbf{j} + 6\mathbf{k}$

b $(-9\mathbf{i} + 15\mathbf{j} + 12\mathbf{k}) \cdot (2.5\mathbf{i} + 15\mathbf{j} + 6\mathbf{k}) = \sqrt{81 + 225 + 144}\sqrt{6.25 + 225 + 26}\cos AHN$

$$\cos AHN = \frac{-22.5 + 225 + 72}{\sqrt{81 + 225 + 144}\sqrt{6.25 + 225 + 26}}$$

$AHN = 37.7^\circ$ (to 1 decimal place)

c $\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AH}$

$= 9\mathbf{i} + \lambda(-9\mathbf{i} + 15\mathbf{j} + 12\mathbf{k})$

You could also use $\frac{1}{3}$ of this direction vector, to get:

$\mathbf{r} = 9\mathbf{i} + \lambda(-3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$

4 **a** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} -4 \\ n-3 \\ -3 \end{pmatrix}$

$\overrightarrow{CB} = \mathbf{b} - \mathbf{c} = \begin{pmatrix} -6 \\ n-9 \\ -1 \end{pmatrix}$

$$\begin{aligned} (-4)^2 + (n-3)^2 + (-3)^2 &= (-6)^2 + (n-9)^2 + (-1)^2 \\ 16 + n^2 - 6n + 9 + 9 &= 36 + n^2 - 18n + 81 + 1 \\ 12n + 34 &= 118 \\ 12n &= 84 \\ n &= 7 \end{aligned}$$

b $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 4 \\ -3 \end{pmatrix}$

$\overrightarrow{CB} = \begin{pmatrix} -6 \\ -2 \\ -1 \end{pmatrix}$

$\overrightarrow{AB} \cdot \overrightarrow{CB} = 24 - 8 + 3 = \sqrt{16+16+9}\sqrt{36+4+1} \cos ABC$

$ABC = \cos^{-1}\left(\frac{19}{\sqrt{41}\sqrt{41}}\right) = 62.4^\circ$ (to 1 decimal place)

5 **a** $$\begin{aligned} (8\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + p\mathbf{k}) &= 0 \\ 8 - 4 + 5p &= 0 \\ 5p &= -4 \\ p &= -\frac{4}{5} \end{aligned}$$

b **i** $\mathbf{r} = -3\mathbf{i} + \mathbf{j} + 5\mathbf{k} + \lambda(7\mathbf{i} - \mathbf{j} - \mathbf{k})$

ii

> When trying to find the intersection of two lines, you will form three equations using just two parameters. The two values you obtain must work in ALL three equations if there is to be an intersection. If the values do not satisfy all three lines, then the lines do not meet.

$-3\mathbf{i} + \mathbf{j} + 5\mathbf{k} + \lambda(7\mathbf{i} - \mathbf{j} - \mathbf{k}) = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + 8\mathbf{j} - 3\mathbf{k})$

$-3 + 7\lambda = 1 + \mu$

$7\lambda - \mu = 4$ .... [1]

$1 - \lambda = -2 + 8\mu$

$\lambda + 8\mu = 3$

$7\lambda + 56\mu = 21$ .... [2]

[2] − [1]:

$58\mu = 17$

$\mu = 4$

$\lambda = 3 - 8(4) = -29$

k-components:

$5 - \lambda = 2 - 3\mu$

$\lambda - 3\mu = 3$

But, using the values obtained above

$\lambda - 3\mu = -29 - 12 \neq 3$

Inconsistent equations, so the lines do not intersect.

6 **a** Using the fact that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{AB}$:

$$\begin{aligned} (-4\mathbf{i} + p\mathbf{j} - 6\mathbf{k}) \cdot (-10\mathbf{i} - 2\mathbf{j} - 10\mathbf{k} + 4\mathbf{i} - p\mathbf{j} + 6\mathbf{k}) &= 0 \\ (-4\mathbf{i} + p\mathbf{j} - 6\mathbf{k}) \cdot (-6\mathbf{i} - (2+p)\mathbf{j} - 4\mathbf{k}) &= 0 \\ 24 - p(2+p) + 24 &= 0 \\ p^2 + 2p - 48 &= 0 \\ (p+8)(p-6) &= 0 \end{aligned}$$

$p = 6 \quad \text{or} \quad p = -8$

But $p > 0$

So $p = 6$

b Rectangle, so

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = -6\mathbf{i} - 8\mathbf{j} - 4\mathbf{k} + (4\mathbf{i} - 6\mathbf{j} + 6\mathbf{k})$$
$$= -2\mathbf{i} - 14\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AC}$$
$$= -4\mathbf{i} + 6\mathbf{j} - 6\mathbf{k} + \lambda(-2\mathbf{i} - 14\mathbf{j} + 2\mathbf{k})$$

c $-4\mathbf{i} + 6\mathbf{j} - 6\mathbf{k} + \lambda(-2\mathbf{i} - 14\mathbf{j} + 2\mathbf{k}) = 3\mathbf{i} + 7\mathbf{j} + \mathbf{k} + \mu(-4\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$

$$-4 - 2\lambda = 3 - 4\mu$$

$2\lambda - 4\mu = -7$ .... [1]

$6 - 14\lambda = 7 - 4\mu$

$14\lambda - 4\mu = -1$ .... [2]

$-6 + 2\lambda = 1 - 3\mu$

$2\lambda + 3\mu = 7$ .... [3]

[2] − [1]:

$12\lambda = 6$

$\lambda = \frac{1}{2}$

[2] then gives:

$7 - 4\mu = -1$

$4\mu = 8$

$\mu = 2$

$2\lambda + 3\mu = 1 + 6 = 7$, which is consistent with equation [3].

The lines intersect at the point with position vector $-4\mathbf{i} + 6\mathbf{j} - 6\mathbf{k} + \frac{1}{2}(-2\mathbf{i} - 14\mathbf{j} + 2\mathbf{k})$

$= -5\mathbf{i} - \mathbf{j} - 5\mathbf{k}$

d $(-2\mathbf{i} - 14\mathbf{j} + 2\mathbf{k}) \cdot (-4\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}) = \sqrt{4 + 196 + 4}\sqrt{16 + 16 + 9}\cos\theta$

$$\cos\theta = \frac{58}{\sqrt{204}\sqrt{41}}$$

$\theta = 50.6^\circ$ (to 1 decimal place)

7 **a** $\overrightarrow{AB} \cdot \overrightarrow{CB} = \begin{pmatrix} -2 \\ -4 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix} = (-2)(2) + (-4)(-4) + (6)(-2) = 0$✔

b $\overrightarrow{AD} = \begin{pmatrix} -10 \\ 20 \\ 10 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$

So $\overrightarrow{AD} = 5\overrightarrow{BC}$

Lines AD and BC are parallel.

c

> The midpoint of points A and B will have position vector $\frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$. That is, its position vector is the mean of the position vectors of A and B.

Position vector of E

$$\overrightarrow{OE} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OD})$$
$$= \frac{1}{2}(4\mathbf{i} + 2\mathbf{j} - \mathbf{k} - 6\mathbf{i} + 22\mathbf{j} + 9\mathbf{k})$$
$$= \frac{1}{2}(-2\mathbf{i} + 24\mathbf{j} + 8\mathbf{k})$$
$$= -\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}$$

Direction

$$\overrightarrow{EC} = 2\mathbf{j} + 7\mathbf{k} - (-\mathbf{i} + 12\mathbf{j} + 4\mathbf{k})$$
$$= \mathbf{i} - 10\mathbf{j} + 3\mathbf{k}$$
$$\mathbf{r} = -\mathbf{i} + 12\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} - 10\mathbf{j} + 3\mathbf{k})$$

8 **a**

$$\overrightarrow{AB} = -7\mathbf{i} + \mathbf{j} + 7\mathbf{k}$$
$$\overrightarrow{CD} = -4\mathbf{i} + 8\mathbf{j} + 12\mathbf{k}$$
$$(-7\mathbf{i} + \mathbf{j} + 7\mathbf{k}) \cdot (-4\mathbf{i} + 8\mathbf{j} + 12\mathbf{k}) = \sqrt{49+1+49}\sqrt{16+64+144}\cos\theta$$
$$\cos\theta = \frac{28+8+84}{\sqrt{99}\sqrt{224}}$$

$\theta = 36.3^\circ$ (to 1 decimal place)_

b $AB: \mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} = -3\mathbf{i} + \mathbf{j} + 8\mathbf{k} + \lambda(-7\mathbf{i} + \mathbf{j} + 7\mathbf{k})$

$CD: \mathbf{r} = \overrightarrow{OC} + \mu\overrightarrow{CD} = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} + \mu(-4\mathbf{i} + 8\mathbf{j} + 12\mathbf{k})$

Intersect if:

$-3\mathbf{i} + \mathbf{j} + 8\mathbf{k} + \lambda(-7\mathbf{i} + \mathbf{j} + 7\mathbf{k}) = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} + \mu(-4\mathbf{i} + 8\mathbf{j} + 12\mathbf{k})$

$-3 - 7\lambda = 5 - 4\mu$

$7\lambda - 4\mu = -8$ .... [1]

$1 + \lambda = -2 + 8\mu$

$\lambda - 8\mu = -3$ .... [2]

$8 + 7\lambda = -2 + 12\mu$

$7\lambda - 12\mu = -10$ .... [3]

[2] + [1]:

$8\lambda - 12\mu = -11$ .... [4]

[4] − [3]:

$\lambda = -1$

Then [1] gives:

$-7 - 4\mu = -8$

$\mu = \frac{1}{4}$

Check:

[1] : $7(-1) - 4\left(\frac{1}{4}\right) = -8$ Yes

[2] : $-1 - 8\left(\frac{1}{4}\right) = -3$ Yes

[3] : $7(-1) - 12\left(\frac{1}{4}\right) = -10$ Yes

Intersect at the point with position vector:

$-3\mathbf{i} + \mathbf{j} + 8\mathbf{k} + \lambda(-7\mathbf{i} + \mathbf{j} + 7\mathbf{k})$

$= -3\mathbf{i} + \mathbf{j} + 8\mathbf{k} - (-7\mathbf{i} + \mathbf{j} + 7\mathbf{k})$

$= 4\mathbf{i} + \mathbf{k}$

c

> The strategy here is to find a vector joining the point E to the line CD. You then need to work out how to make sure that this vector is perpendicular to the original line. Once you have done that, you find the magnitude of the joining vector.

Equation of CD:

$\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} + \mu(-4\mathbf{i} + 8\mathbf{j} + 12\mathbf{k})$

Any point on CD will have a position vector that can be written in this form.

Let N be the foot of the perpendicular from E to CD. N lies on the line, so

$$\mathbf{N} = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} + \mu(-4\mathbf{i} + 8\mathbf{j} + 12\mathbf{k})$$
$$\overrightarrow{EN} = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} + \mu(-4\mathbf{i} + 8\mathbf{j} + 12\mathbf{k}) - (5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$
$$= -4\mu\mathbf{i} + (8\mu - 5)\mathbf{j} + (12\mu - 6)\mathbf{k}$$

For shortest distance, $\overrightarrow{EN}$ must be perpendicular to the line CD.

$$
\begin{aligned}
\overrightarrow{EN}\cdot\overrightarrow{CD} &= 0\\
(-4\mu\mathbf{i}+(8\mu-5)\mathbf{j}+(12\mu-6)\mathbf{k}).(-4\mathbf{i}+8\mathbf{j}+12\mathbf{k}) &= 0\\
16\mu+8(8\mu-5)+12(12\mu-6) &= 0\\
224\mu-40-72 &= 0
\end{aligned}
$$

$$
\begin{aligned}
\mu &= \frac{112}{224}=\frac{1}{2}\\
\overrightarrow{EN} &= -4\left(\frac{1}{2}\right)\mathbf{i}+(8\left(\frac{1}{2}\right)-5)\mathbf{j}\\
&\quad +(12\left(\frac{1}{2}\right)-6)\mathbf{k}\\
&= -2\mathbf{i}-\mathbf{j}\\
\left|\overrightarrow{EN}\right| &= \sqrt{4+1}=\sqrt{5}
\end{aligned}
$$

9 **a** $\overrightarrow{PQ}=\begin{pmatrix}-0.5-9\\6-2\\6.5-4\end{pmatrix}=\begin{pmatrix}-9.5\\4\\2.5\end{pmatrix}$

$\overrightarrow{PS}=\begin{pmatrix}4.5-9\\-4-2\\-3.5-4\end{pmatrix}=\begin{pmatrix}-4.5\\-6\\-7.5\end{pmatrix}$

b $\overrightarrow{OR}=\overrightarrow{OQ}+\overrightarrow{QR}$

Rhombus, so $\overrightarrow{QR}=\overrightarrow{PS}$

$\overrightarrow{OR}=\overrightarrow{OQ}+\overrightarrow{PS}$

$=\begin{pmatrix}-0.5\\6\\6.5\end{pmatrix}+\begin{pmatrix}-4.5\\-6\\-7.5\end{pmatrix}=\begin{pmatrix}-5\\0\\-1\end{pmatrix}$

$R(-5,0,-1)$

c $\overrightarrow{PQ}\cdot\overrightarrow{PS}=-9.5\times-4.5+4\times-6+2.5\times-7.5$

$=0$

PQ is perpendicular to PS, so the shape is a rectangle.

$\left|\overrightarrow{PQ}\right|=\sqrt{9.5^2+4^2+2.5^2}=\sqrt{112.5}$

$\left|\overrightarrow{PS}\right|=\sqrt{4.5^2+6^2+7.5^2}=\sqrt{112.5}$

Perpendicular sides have the same length, so it is a square.

d

> You will notice that column vectors are generally easier to write quickly. It is perfectly acceptable to convert vectors written using **i**, **j** and **k** to column vectors.

Midpoint of QS has position vector

$\frac{1}{2}\left(\overrightarrow{OQ}+\overrightarrow{OS}\right)=\frac{1}{2}\begin{pmatrix}4\\2\\3\end{pmatrix}$

$T(2,1,1.5)$

e **i** For example

$\mathbf{r}=\mathbf{v}+\lambda(\mathbf{t}-\mathbf{v})$

$\mathbf{r}=\begin{pmatrix}5\\17.5\\-13.5\end{pmatrix}+\lambda\begin{pmatrix}-3\\-16.5\\15\end{pmatrix}$

ii $\overrightarrow{VT}\cdot\overrightarrow{PR}=\begin{pmatrix}-3\\-16.5\\15\end{pmatrix}\cdot\begin{pmatrix}-14\\-2\\-5\end{pmatrix}=42+33-75=0$

VT is perpendicular to PR and V lies on PR.

So V is the foot of the perpendicular from V to PR.

iii The point V is a point on a line through the centre of the square and perpendicular to the square.

$VPQRS$ is a right, square-based pyramid with V as the vertex.

10 **a** $\lambda = -1$

$$\begin{pmatrix} 3 \\ 9 \\ 2 \end{pmatrix} - 1\begin{pmatrix} 3 \\ -10 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 19 \\ -1 \end{pmatrix} = \mathbf{p}$$

b $|\overrightarrow{PQ}| = \left|\begin{pmatrix} -6 \\ 7 \\ 10 \end{pmatrix}\right| = \sqrt{36+49+100} = \sqrt{185}$

c $\begin{pmatrix} -6 \\ 7 \\ -10 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -10 \\ 3 \end{pmatrix} = \sqrt{36+49+100}\sqrt{9+100+9}\cos\theta$

$$\cos\theta = \frac{-18-70-30}{\sqrt{185}\sqrt{118}}$$

$\theta = 143.0^\circ$ (to 1 decimal place)

d Foot of perpendicular, N, is at $(-3, 29, -4)$

$\overrightarrow{QN} = (-3, 29, -4) - (-6, 26, -11)$

$\overrightarrow{QN} = (3, 3, 7)$

$|\overrightarrow{QN}| = \sqrt{9+9+49} = \sqrt{67}$

11 **a** $\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$

$$= \begin{pmatrix} 7 \\ 1 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$$

b P lies on AB, so its position vector can be written in the form:

$$\mathbf{p} = \begin{pmatrix} 7 \\ 1 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$$

$$\overrightarrow{QP} = \begin{pmatrix} 7 \\ 1 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} 7+3\lambda \\ 6+4\lambda \\ -1-5\lambda \end{pmatrix}$$

QP is perpendicular to AB, so:

$$\begin{pmatrix} 7+3\lambda \\ 6+4\lambda \\ -1-5\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} = 0$$

$$21 + 9\lambda + 24 + 16\lambda + 5 + 25\lambda = 0$$

$$50\lambda = -50$$

$$\lambda = -1$$

$$\overrightarrow{QP} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$$

This passes through Q, so the vector equation of PQ is

$\mathbf{r} = \overrightarrow{OQ} + \lambda\overrightarrow{QP}$

$$\mathbf{r} = \begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$$

c

Note that P lies on AB and PQ is perpendicular to AB. So, P is the foot of the perpendicular to the line AB from the point Q. The magnitude of QP is, therefore, the shortest distance required.

$$\left|\begin{pmatrix}4\\2\\4\end{pmatrix}\right| = \sqrt{16+4+16} = \sqrt{36} = 6$$

12 **a** $\mathbf{r} = \begin{pmatrix}3\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}4\\2\\3\end{pmatrix}$

$\lambda = 0$ gives:

$$\mathbf{r} = \begin{pmatrix}3\\2\\5\end{pmatrix} = \begin{pmatrix}3\\p\\5\end{pmatrix}$$

$p = 2$

$\lambda = -1$ gives:

$$\mathbf{r} = \begin{pmatrix}3\\2\\5\end{pmatrix} - \begin{pmatrix}4\\2\\3\end{pmatrix} = \begin{pmatrix}q\\0\\2\end{pmatrix}$$

$q = -1$

b The lines intersect when:

$$\begin{pmatrix}3\\2\\5\end{pmatrix} + s\begin{pmatrix}4\\2\\3\end{pmatrix} = \begin{pmatrix}0\\3\\1\end{pmatrix} + t\begin{pmatrix}7\\1\\7\end{pmatrix}$$

$3 + 4s = 7t$

$4s - 7t = -3$ .... [1]

$2 + 2s = 3 + t$

$2s - t = 1$ .... [2]

$5 + 3s = 1 + 7t$

$3s - 7t = -4$ .... [3]

7[2] − [1]:

$10s = 10$

$s = 1$

Then [2] gives:

$2 - t = 1$

$t = 1$

$3s - 7t = 3 - 7 = -4$

So, $s = 1$ and $t = 1$ satisfy all three equations.

So the lines intersect at the point with position vector:

$$\begin{pmatrix}3\\2\\5\end{pmatrix} + s\begin{pmatrix}4\\2\\3\end{pmatrix} = \begin{pmatrix}3\\2\\5\end{pmatrix} + \begin{pmatrix}4\\2\\3\end{pmatrix} = \begin{pmatrix}7\\4\\8\end{pmatrix}$$

c

> The angle between two lines is the angle between their direction vectors.

$$\begin{pmatrix}4\\2\\3\end{pmatrix} \cdot \begin{pmatrix}7\\1\\7\end{pmatrix} = \sqrt{16+4+9}\sqrt{49+1+49}\cos\theta$$

$$\cos\theta = \frac{28+2+21}{\sqrt{29}\sqrt{99}}$$

$\theta = 17.9^\circ$ (to 1 decimal place)

Chapter 10
Differential equations

P3 **This chapter is for Pure Mathematics 3 students only.**

EXERCISE 10A

1 **c**

$$s^2\frac{ds}{dt} = \cos\ (t+5)$$
$$\int s^2\frac{ds}{dt}\,dt = \int \cos\,(t+5)\,dt$$
$$\int s^2\,ds = \int \cos\,(t+5)\,dt$$
$$\frac{1}{3}s^3 = \sin(t+5)+c$$

Remember that you do not need to include the constant of integration on both sides.

e

$$\frac{dy}{dx} = \frac{y+1}{x}$$
$$\frac{1}{y+1}\frac{dy}{dx} = \frac{1}{x}$$
$$\int \frac{1}{y+1}\frac{dy}{dx}\,dx = \int \frac{1}{x}\,dx$$
$$\int \frac{1}{y+1}\,dy = \int \frac{1}{x}\,dx$$
$$\ln|y+1| = \ln|x| + c$$

If $y = -1$, then $\frac{dy}{dx} = 0$ and $\frac{y+1}{x} = 0$.

So $y = -1$ is also a solution to the differential equation.

2 **a**

$$\cos^2 x\frac{dy}{dx} = y$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos^2 x}$$
$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{\cos^2 x}\,dx$$
$$\int \frac{1}{y}\,dy = \int \sec^2 x\,dx$$
$$\ln|y| = \tan x + c$$

Using the fact that $y = 5$ when $x = 0$:

$$\ln\ 5 = \tan 0 + c$$
$$c = \ln\ 5$$
$$\ln|y| = \tan x + \ln\ 5$$
$$\ln\left|\frac{y}{5}\right| = \tan x$$
$$\left|\frac{y}{5}\right| = e^{\tan x}$$
$$e^{\tan x} > 0$$
$$y = 5e^{\tan x}$$

You should always consider when it might be possible to remove the modulus signs. Given that any positive number to any power will give a positive result, you can often remove modulus signs when your solution is in exponential form.

b
$$xy\frac{dy}{dx} = \frac{x^3 - x}{1 - \sqrt{y}}$$
$$y\left(1 - \sqrt{y}\right)\frac{dy}{dx} = \frac{x^3 - x}{x}$$
$$\int y\left(1 - \sqrt{y}\right)\frac{dy}{dx}\,dx = \int \frac{x^3 - x}{x}\,dx$$
$$\int \left(y - y^{\frac{3}{2}}\right)dy = \int \left(x^2 - 1\right)dx$$
$$\frac{1}{2}y^2 - \frac{2}{5}y^{\frac{5}{2}} = \frac{1}{3}x^3 - x + c$$

Using the fact that $y = 4$ when $x = 3$:

$$8 - \frac{2}{5}(32) = 9 - 3 + c$$
$$c = 8 - \frac{64}{5} - 6 = -\frac{54}{5}$$
$$\frac{1}{2}y^2 - \frac{2}{5}y^{\frac{5}{2}} = \frac{1}{3}x^3 - x - \frac{54}{5}$$

3
$$\frac{dy}{dx} = e^{x-y}$$
$$\frac{dy}{dx} = e^x e^{-y}$$
$$e^y\frac{dy}{dx} = e^x$$
$$\int e^y\frac{dy}{dx}\,dx = \int e^x\,dx$$
$$\int e^y\,dy = \int e^x\,dx$$
$$e^y = e^x + c$$

Using the fact that $y = \ln 2$ when $x = 0$:

$$e^{\ln 2} = e^0 + c$$
$$2 = 1 + c$$
$$c = 1$$
$$e^y = e^x + 1$$
$$y = \ln(e^x + 1)$$

Remember the basic laws of indices. In particular, $x^{a+b} = x^a x^b$.

4 **a** $\frac{dy}{dx} = 5y^3x$ because $\frac{dy}{dx}$ is the gradient function.

b
$$\frac{1}{y^3}\frac{dy}{dx} = 5x$$
$$\int \frac{1}{y^3}\frac{dy}{dx}\,dx = \int 5x\,dx$$
$$\int y^{-3}\,dy = \frac{5x^2}{2} + c$$
$$-\frac{1}{2}y^{-2} = \frac{5x^2}{2} + c$$

Using the fact that $y = 1$ when $x = 1$:

$$-\frac{1}{2}=\frac{5}{2}+c$$
$$c=-3$$
$$-\frac{1}{2}y^{-2}=\frac{5x^2}{2}-3$$
$$-\frac{1}{2y^2}=\frac{5x^2-6}{2}$$
$$2y^2=-\frac{2}{5x^2-6}$$
$$y^2=-\frac{1}{5x^2-6}$$
$$y^2=\frac{1}{6-5x^2}$$

5 **a** $$\frac{1}{(2-x)(x+1)}=\frac{A}{2-x}+\frac{B}{x+1}$$
$$1=A(x+1)+B(2-x)$$

Letting $x=-1$:

$$1=0+3B$$
$$B=\frac{1}{3}$$

Letting $x=2$:

$$1=3A+0$$
$$A=\frac{1}{3}$$
$$\frac{1}{(2-x)(x+1)}=\frac{1}{3(2-x)}+\frac{1}{3(x+1)}$$

Remind yourself about partial fractions. These were covered in Chapter 7.

b **i** $$\frac{dx}{dt}=(2-x)(x+1)$$
$$\frac{1}{(2-x)(x+1)}\frac{dx}{dt}=1$$
$$\int\frac{1}{(2-x)(x+1)}\frac{dx}{dt}\,dt=\int 1\,dt$$
$$\int\left(\frac{1}{3(2-x)}+\frac{1}{3(x+1)}\right)dx=t+c$$
$$-\frac{1}{3}\ln|2-x|+\frac{1}{3}\ln|x+1|=t+c$$

Using the fact that $t=0$ when $x=0$:

$$-\frac{1}{3}\ln 2+\frac{1}{3}\ln 1=0+c$$
$$c=-\frac{1}{3}\ln 2$$
$$-\frac{1}{3}\ln|2-x|+\frac{1}{3}\ln|x+1|=t+-\frac{1}{3}\ln 2$$
$$\ln|x+1|-\ln|2-x|+\ln 2=3t$$
$$\ln\left|\frac{2(x+1)}{2-x}\right|=3t$$
$$\frac{2(x+1)}{2-x}=e^{3t}$$
$$2x+2=2e^{3t}-xe^{3t}$$
$$x\left(2+e^{3t}\right)=2e^{3t}-2$$
$$x=\frac{2\left(e^{3t}-1\right)}{e^{3t}+2}$$

ii As t becomes large:

$$
\begin{aligned}
x &= \frac{2\left(e^{3t}-1\right)}{e^{3t}+2} \\
&= \frac{2\left(1-\frac{1}{e^{3t}}\right)}{\left(1+\frac{2}{e^{3t}}\right)} \\
&\longrightarrow \frac{2\,(1-0)}{(1+0)} = 2
\end{aligned}
$$

6

$$
\begin{aligned}
\frac{dw}{dt} &= 0.001(100-w)^2 \\
\frac{1}{(100-w)^2}\frac{dw}{dt} &= 0.001 \\
\int (100-w)^{-2}\frac{dw}{dt}\,dt &= \int 0.001\,dt \\
\int (100-w)^{-2}\,dw &= \int 0.001\,dt \\
(100-w)^{-1} &= 0.001t + c \\
\frac{1}{100-w} &= \frac{t}{1000} + c
\end{aligned}
$$

Using the fact that $t = 0$ when $w = 0$:

$$
\begin{aligned}
\frac{1}{100} &= 0 + c \\
\frac{1}{100-w} &= \frac{t}{1000} + \frac{1}{100} \\
\frac{1}{100-w} &= \frac{t+10}{1000} \\
100 - w &= \frac{1000}{t+10} \\
w &= 100 - \frac{1000}{t+10}
\end{aligned}
$$

7

$$
\begin{aligned}
4\frac{dx}{dt} &= \left(3x^2 - x\right)\cos t \\
\frac{4}{x(3x-1)}\frac{dx}{dt} &= \cos t
\end{aligned}
$$

At this point you need to spot that partial fractions are required to integrate the left-hand side.

$$
\begin{aligned}
\frac{4}{x(3x-1)} &= \frac{A}{x} + \frac{B}{3x-1} \\
4 &= A(3x-1) + Bx
\end{aligned}
$$

Letting $x = 0$:

$$
\begin{aligned}
4 &= -A \\
A &= -4
\end{aligned}
$$

Letting $x = \frac{1}{3}$:

$$
\begin{aligned}
4 &= 0 + \frac{1}{3}B \\
B &= 12
\end{aligned}
$$

So the equation can be written as:

$$
\begin{aligned}
\left(\frac{12}{3x-1} - \frac{4}{x}\right)\frac{dx}{dt} &= \cos t \\
\int \left(\frac{12}{3x-1} - \frac{4}{x}\right)\frac{dx}{dt}\,dt &= \int \cos t\,dt \\
4\ln|3x-1| - 4\ln|x| &= \sin t + c \\
4\ln\left|\frac{3x-1}{x}\right| &= \sin t + c
\end{aligned}
$$

Using the fact that $x = 0.4$ when $t = 0$:

$$4\ \ln\left(\frac{1}{2}\right) = 0 + c$$

$$4\ \ln\left|\frac{3x-1}{x}\right| = \sin t + 4\ \ln\left(\frac{1}{2}\right)$$

$$4\ \ln\left|\frac{2\,(3x-1)}{x}\right| = \sin t$$

$$\ln\left|\frac{2\,(3x-1)}{x}\right| = \frac{1}{4}\sin t$$

$$\frac{2\,(3x-1)}{x} = \mathrm{e}^{\frac{1}{4}\sin t}$$

$$6x - 2 = x\mathrm{e}^{\frac{1}{4}\sin t}$$

$$x\left(6 - \mathrm{e}^{\frac{1}{4}\sin t}\right) = 2$$

$$x = \frac{2}{6 - \mathrm{e}^{\frac{1}{4}\sin t}}$$

8

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2-1}{\mathrm{e}^{y+1}}$$

$$y\mathrm{e}^{y+1}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2-1}{x}$$

$$\int y\mathrm{e}^{y+1}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{x^2-1}{x}\,\mathrm{d}x$$

$$\int y\mathrm{e}^{y+1}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \left(x - \frac{1}{x}\right)\mathrm{d}x$$

Using integration by parts:

$$y\mathrm{e}^{y+1} - \int 1 \times \mathrm{e}^{y+1}\,\mathrm{d}x = \frac{1}{2}x^2 - \ln|x| + c$$

$$y\mathrm{e}^{y+1} - \mathrm{e}^{y+1} = \frac{1}{2}x^2 - \ln|x| + c$$

Remember that $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$.

9

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+y^2}{4+x^2}$$

$$\frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{4+x^2}$$

$$\int \frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{1}{4+x^2}\,\mathrm{d}x$$

$$\tan^{-1} y = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

$$y = \tan\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c\right)$$

Using the fact that $y = 1$ when $x = 0$:

$$1 = \tan\left(\frac{1}{2}(0) + c\right)$$

$$c = \frac{\pi}{4}$$

$$y = \tan\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{\pi}{4}\right)$$

Remember that $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$.

10

$$x^3\frac{\mathrm{d}y}{\mathrm{d}x}+2y=1$$
$$x^3\frac{\mathrm{d}y}{\mathrm{d}x}=1-2y$$
$$\frac{1}{1-2y}\frac{\mathrm{d}y}{\mathrm{d}x}=x^{-3}$$
$$\int\frac{1}{1-2y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x=\int x^{-3}\,\mathrm{d}x$$
$$-\frac{1}{2}\ln|1-2y|=-\frac{1}{2}x^{-2}+c$$
$$\ln|1-2y|=\frac{1}{x^2}+B$$

11

$$\frac{\mathrm{d}x}{\mathrm{d}t}=4x\cos^2 t$$
$$\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}=4\cos^2 t$$
$$\int\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t=\int 4\cos^2 t\,\mathrm{d}t$$
$$\ln|x|=\int 4\left(\frac{1}{2}+\frac{1}{2}\cos 2t\right)\mathrm{d}t$$
$$\ln|x|=\int(2+2\cos 2t)\,\mathrm{d}t$$
$$\ln|x|=2t+\sin 2t+c$$
$$x=\mathrm{e}^{2t+\sin 2t+c}$$

Using the fact that $x=1$ when $t=0$:

$1=\mathrm{e}^c$

$c=0$

$x=\mathrm{e}^{2t+\sin 2t}$

In this solution you need to use the fact that $\cos 2A=2\cos^2 A-1$, which gives $\cos^2 A=\frac{1}{2}+\frac{1}{2}\cos 2A$.

12 **a**

$$u=t^2$$
$$\frac{\mathrm{d}u}{\mathrm{d}t}=2t$$
$$\mathrm{d}u=2t\,\mathrm{d}t$$
$$\frac{1}{2}\mathrm{d}u=t\,\mathrm{d}t$$
$$\int t\mathrm{e}^{t^2}\,\mathrm{d}t=\int\mathrm{e}^u\left(\frac{1}{2}\right)\mathrm{d}u$$
$$=\frac{1}{2}\mathrm{e}^u+c$$
$$=\frac{1}{2}\mathrm{e}^{t^2}+c$$

b

Always watch out for opportunities to use the answer from the first part in later parts of any question.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{t\left(\mathrm{e}^{t^2}+5\right)}{x^2}$$
$$x^2\frac{\mathrm{d}x}{\mathrm{d}t}=t\mathrm{e}^{t^2}+5t$$
$$\int x^2\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t=\int t\mathrm{e}^{t^2}+5t\,\mathrm{d}t$$

So, using the result from part **a**:

$$\frac{1}{3}x^3=\frac{1}{2}\mathrm{e}^{t^2}+\frac{5t^2}{2}+c$$

Using the fact that $x=1$ when $t=0$:

$$\frac{1}{3} = \frac{1}{2} + c$$

$$c = -\frac{1}{6}$$

$$\frac{1}{3}x^3 = \frac{1}{2}e^{t^2} + \frac{5t^2}{2} - \frac{1}{6}$$

13

> Remember that if two quantities x and y are directly proportional then $y = kx$ for some constant k.

$$\frac{dy}{dx} = kxy^2$$

When $x = 1$ and $y = 3$, $\frac{dy}{dx} = 6$:

$$6 = 9k$$

$$k = \frac{2}{3}$$

$$\frac{dy}{dx} = kxy^2$$

$$y^{-2}\frac{dy}{dx} = kx$$

$$\int y^{-2}\frac{dy}{dx}\,dx = \int kx\,dx$$

$$-y^{-1} = \frac{1}{2}kx^2 + c$$

Using the fact that $y = 3$ when $x = 1$:

$$-\frac{1}{3} = \frac{1}{2}k + c = \frac{1}{3} + c$$

$$c = -\frac{2}{3}$$

$$-\frac{1}{y} = \frac{1}{3}x^2 - \frac{2}{3} = \frac{x^2 - 2}{3}$$

$$y = -\frac{3}{x^2 - 2} = \frac{3}{2 - x^2}$$

14

$$\left(3x + 2x^3\right)\frac{dx}{dt} = k\left(3x^2 + x^4\right)$$

$$\frac{3x + 2x^3}{3x^2 + x^4}\frac{dx}{dt} = k$$

$$\int \frac{3x + 2x^3}{3x^2 + x^4}\frac{dx}{dt}\,dt = \int k\,dt$$

$$\frac{1}{2}\int \frac{6x + 4x^3}{3x^2 + x^4}\frac{dx}{dt}\,dt = \int k\,dt$$

> Notice that this now means the is the derivative of the numerator, so you can use logarithms to integrate.

$$\frac{1}{2}\ln\left|3x^2 + x^4\right| = kt + c$$

$$\frac{1}{2}\ln\left(3x^2 + x^4\right) = kt + c$$

Using the fact that $t = 0$ when $x = 1$:

$$\frac{1}{2}\ln 4 = 0 + c$$

$$c = \ln 2$$

$$\frac{1}{2}\ln(3x^2 + x^4) = kt + \ln 2$$

Now using the fact that $t = \frac{1}{2}$ when $x = 2$:

$$\frac{1}{2}\ln 28 = \frac{1}{2}k + \ln 2$$
$$\frac{1}{2}k = \frac{1}{2}\ln 28 - \frac{1}{2}\ln 4 = \frac{1}{2}\ln 7$$
$$k = \ln 7$$
$$\frac{1}{2}\ln\left(3x^2 + x^4\right) = t\ln 7 + \ln 2$$
$$\ln\left(3x^2 + x^4\right) = 2t\ln 7 + 2\ln 2$$
$$\ln 7^{2t} = \ln\left(\frac{3x^2 + x^4}{4}\right)$$
$$7^{2t} = \frac{3x^2 + x^4}{4}$$

15 **a**

$$u = \ln x$$
$$\frac{du}{dx} = \frac{1}{x}$$
$$\frac{1}{x}\,dx = du$$
$$\int \frac{1}{x\ln x}\,dx$$
$$= \int \frac{1}{u}\,du$$
$$= \ln|u| + c$$
$$= \ln|\ln x| + c$$

b

$$\ln x\frac{dy}{dx} = \frac{y}{x}$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x\ln x}$$
$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{x\ln x}\,dx$$

Using the result from part **a**:

$$\ln|y| = \ln|\ln x| + c$$
$$|y| = e^{\ln|\ln x|}e^c$$
$$y > 0$$
$$y = be^{\ln|\ln x|}$$

where $b = e^c$

16

$$\frac{dm}{dt} = -0.05m$$
$$\frac{1}{m}\frac{dm}{dt} = -0.05$$
$$\int \frac{1}{m}\frac{dm}{dt}\,dt = -\int 0.05\,dt$$
$$\ln|m| = -0.05t + c$$

Using the fact that $m = 65$ when $t = 0$:

$$\ln 65 = c$$
$$m > 0$$
$$\ln m = -0.05t + \ln 65$$
$$m = e^{-0.05t + \ln 65}$$
$$= e^{-0.05t}e^{\ln 65}$$
$$= 65e^{-0.05t}$$

When the mass remaining is 6 mg:

$$65e^{-0.05t} = 6$$
$$e^{-0.05t} = \frac{6}{65}$$
$$-0.05t = \ln\frac{6}{65}$$
$$t = -20\ln\frac{6}{65} = 48 \text{ hours (to the nearest hour)}$$

EXERCISE 10B

1 **a** In part **a** the change is a decrease, so the rate of change is negative.

$$\frac{dh}{dt} = -kh^2, k > 0$$

b In part **b** the change is an increase, so the rate of change is positive.

$$\frac{dn}{dt} = kn, k > 0$$

c In part **c** the change is a decrease, so the rate of change is negative.

$$\frac{dv}{dt} = -kv(v+1), k > 0$$

d In part **d** the change is a decrease, so the rate of change is negative.

$$\frac{dV}{dt} = -kV, k > 0$$

e In part **e** the change is an increase, so the rate of change is positive.

$$\frac{dC}{dt} = kC^3, k > 0$$

f If two quantities, x and y, are inversely proportional then $y = \frac{k}{x}$.

$$\frac{dv}{dt} = \frac{k}{e^{\frac{t}{2}}}$$

2 **a** $\frac{dx}{dt} = -k\sqrt{x}, k > 0$

b
$$\frac{dx}{dt} = -k\sqrt{x}$$
$$x^{-\frac{1}{2}}\frac{dx}{dt} = -k$$
$$\int x^{-\frac{1}{2}}\frac{dx}{dt}\,dt = \int -k\,dt$$
$$2x^{\frac{1}{2}} = -kt + c$$

Using the fact that $x = 6.25$ when $t = 0$:

$$2\sqrt{6.25} = 0 + c$$
$$c = 5$$
$$2x^{\frac{1}{2}} = -kt + 5$$

Using the fact that $x = 4$ when $t = 120$:

$$2\sqrt{4} = -120k + 5$$
$$k = \frac{1}{120}$$
$$2x^{\frac{1}{2}} = -\frac{1}{120}t + 5$$
$$-240\sqrt{x} = t - 600$$
$$t = 600 - 240\sqrt{x}$$

c half $= 3.125$

$$t = 600 - 240\sqrt{3.125}$$
$$= 176 \text{ seconds (to 3 significant figures)}$$

3 **a** $\frac{\mathrm{d}A}{\mathrm{d}t} = kA, k > 0$

b $$\frac{\mathrm{d}A}{\mathrm{d}t} = kA$$
$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} = k$$

Note that $A > 0$.

$$\int \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t}\,\mathrm{d}t = \int k\,\mathrm{d}t$$
$$\ln A = kt + c$$
$$A = \mathrm{e}^{kt+c} = A_0\mathrm{e}^{kt}$$

Using the fact that $A = 12$ when $t = 0$:

$$12 = A_0\mathrm{e}^0$$
$$A = 12\mathrm{e}^{kt}$$

> If you know that you will be taking logarithms of positive numbers then there is no need to use the modulus function.

c $A = 12\mathrm{e}^{kt}$

Using the fact that $A = 48$ when $t = 25$:

$$48 = 12\mathrm{e}^{25k}$$
$$\mathrm{e}^{25k} = 4$$
$$25k = \ln 4$$
$$k = \frac{1}{25}\ln 4$$
$$A = 12\mathrm{e}^{\left(\frac{1}{25}\ln 4\right)t}$$

When $t = 60$:

$$A = 12\mathrm{e}^{\left(\frac{1}{25}\ln 4\right)60} = \$334 \text{ per kg (to 3 significant figures)}$$

4 **a** $\frac{\mathrm{d}x}{\mathrm{d}t} = k(100 - x), k > 0$

b $$\frac{\mathrm{d}x}{\mathrm{d}t} = k(100 - x)$$
$$\frac{1}{100 - x}\frac{\mathrm{d}x}{\mathrm{d}t} = k$$
$$\int \frac{1}{100 - x}\frac{\mathrm{d}x}{\mathrm{d}t} = \int k\,\mathrm{d}t$$
$$-\ln|100 - x| = kt + c$$

Using the fact that $x = 25$ when $t = 0$:

$$-\ln 75 = c$$
$$-\ln|100 - x| = kt - \ln 75$$
$$\ln|100 - x| = \ln 75 - kt$$
$$100 - x = \mathrm{e}^{\ln 75 - kt} = 75\mathrm{e}^{-kt}$$
$$x = 100 - 75\mathrm{e}^{-kt}$$

c $x = 100 - 75\mathrm{e}^{-kt}$

Using the fact that $x = 85$ when $t = 180$:

$$
\begin{aligned}
85 &= 100 - 75\mathrm{e}^{-180k} \\
75\mathrm{e}^{-180k} &= 15 \\
\mathrm{e}^{-180k} &= \frac{1}{5} \\
-180k &= \ln\frac{1}{5} = -\ln 5 \\
180k &= \ln 5 \\
k &= \frac{1}{180}\ln 5 \\
x &= 100 - 75\mathrm{e}^{-\left(\frac{1}{180}\ln 5\right)t}
\end{aligned}
$$

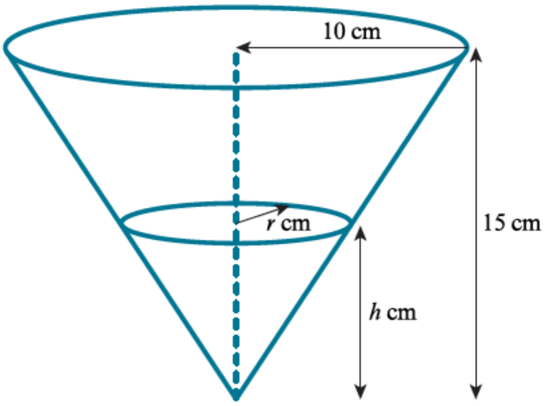
When t = 195:

$$x = 100 - 75\mathrm{e}^{-\left(\frac{1}{180}\ln 5\right)(195)} = 86.9^\circ\mathrm{C}$$

d $x = 100 - 75\mathrm{e}^{-\left(\frac{1}{180}\ln 5\right)t} \to 100 - 75 \times 0$

$= 100^\circ\mathrm{C}$

5 **a**



Using similar triangles:

$$
\begin{aligned}
\frac{r}{10} &= \frac{h}{15} \\
r &= \frac{2}{3}h
\end{aligned}
$$

$$
\begin{aligned}
V &= \frac{1}{3}\pi r^2 h \\
&= \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2 h \\
&= \frac{1}{3}\pi\left(\frac{4}{9}\right)h^3 \\
V &= \frac{4}{27}\pi h^3 \\
\frac{\mathrm{d}V}{\mathrm{d}h} &= \frac{4}{27}(3\pi h^2) = \frac{4}{9}\pi h^2
\end{aligned}
$$

Remember that the volume of a right cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.

b Using the chain rule:

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}V}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t} \\
-16 &= \frac{4}{9}\pi h^2 \times \frac{\mathrm{d}h}{\mathrm{d}t} \\
\frac{\mathrm{d}h}{\mathrm{d}t} &= -\frac{144}{4\pi h^2} = -\frac{36}{\pi h^2}
\end{aligned}
$$

c

$$\begin{aligned}\frac{dh}{dt} &= -\frac{36}{\pi h^2}\\ \pi h^2\frac{dh}{dt} &= -36\\ \int \pi h^2\frac{dh}{dt}\,dt &= \int -36\,dt\\ \frac{1}{3}\pi h^3 &= -36t + c\\ t &= -\frac{1}{36}\left(\frac{1}{3}\pi h^3 - c\right)\end{aligned}$$

Using the fact that $h = 15$ when $t = 0$:

$$\begin{aligned}0 &= -\frac{1}{36}\left(\frac{1}{3}\pi 15^3 - c\right)\\ c &= \frac{1}{3}\pi 15^3 = 1125\pi\\ t &= -\frac{1}{36}\left(\frac{1}{3}\pi h^3 - 1125\pi\right)\\ &= \frac{\pi}{36}\left(1125 - \frac{h^3}{3}\right)\end{aligned}$$

6 **a** $\frac{dP}{dt} = kP, k > 0$

b

$$\begin{aligned}\frac{1}{P}\frac{dP}{dt} &= k\\ \int \frac{1}{P}\frac{dP}{dt}\,dt &= \int k\,dt\\ \ln P &= kt + c\\ P &= e^{kt+c}\\ P &= Ae^{kt}\end{aligned}$$

Using the fact that $P = P_0$when $t = 0$:

$$\begin{aligned}P_0 &= Ae^0 = A\\ P &= P_0e^{kt}\end{aligned}$$

c Using the fact that $P = 1.5P$ when $t = 2$:

$$\begin{aligned}1.5P_0 &= P_0e^{2k}\\ e^{2k} &= 1.5\\ 2k &= \ln 1.5\\ k &= \frac{1}{2}\ln 1.5\end{aligned}$$

$$\begin{aligned}P &= P_0e^{\left(\frac{1}{2}\ln 1.5\right)t}\\ P &= 3P_0\text{when}\\ 3P_0 &= P_0e^{\left(\frac{1}{2}\ln 1.5\right)t}\\ e^{\left(\frac{1}{2}\ln 1.5\right)t} &= 3\\ \left(\frac{1}{2}\ln 1.5\right)t &= \ln 3\end{aligned}$$

$$t = \frac{\ln 3}{\left(\frac{1}{2}\ln 1.5\right)} = 5.42 \text{ minutes (to 3 significant figures)}$$

7 **a** $\frac{dr}{dt} = \frac{k}{\sqrt{r}}$

Using the fact that $\frac{dr}{dt} = 1.4$ when $r = 7.84$:

$$\begin{aligned}1.4 &= \frac{k}{\sqrt{7.84}}\\ k &= 1.4\sqrt{7.84} = 3.92\\ \frac{dr}{dt} &= \frac{3.92}{\sqrt{r}}\end{aligned}$$

b
$$\frac{dr}{dt} = \frac{3.92}{\sqrt{r}}$$
$$r^{\frac{1}{2}}\frac{dr}{dt} = 3.92$$
$$\int r^{\frac{1}{2}}\frac{dr}{dt}\,dt = \int 3.92\,dt$$
$$\frac{2}{3}r^{\frac{3}{2}} = 3.92t + c$$
Using the fact that $r = 7.84$ when $t = 4$:
$$\frac{2}{3}(21.952) = 15.68 + c$$
$$c = -1.045$$
$$\frac{2}{3}r^{\frac{3}{2}} = 3.92t - 1.045...$$
$$r^{\frac{3}{2}} = 5.88t - 1.568$$
$$r = \sqrt[3]{(5.88t - 1.568)^2}$$

c At the start, $t = 0$:
$$r = \sqrt[3]{(0 - 1.568)^2}$$
$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi(0 - 1.568)^2$$
$$= 10.3\text{ cm}^3 \text{ (to 3 significant figures)}$$

8 T = temperatures

t = time
$$\frac{dT}{dt} = -k(T - 7)$$
$$\frac{1}{T-7}\frac{dT}{dt} = -k$$
$$\int \frac{1}{T-7}\frac{dT}{dt}\,dt = \int -k\,dt$$
$$\ln|T - 7| = -kt + c$$
Using the fact that $T = 94$ when $t = 0$:
$$\ln 87 = c$$
$$\ln|T - 7| = -kt + \ln 87$$
$$T - 7 = e^{-kt+\ln 87}$$
$$T = 7 + 87e^{-kt}$$
Using the fact that $T = 54$ when $t = 10$:
$$54 = 7 + 87e^{-10k}$$
$$e^{-10k} = \frac{47}{87}$$
$$-10k = \ln\frac{47}{87}$$
$$k = -\frac{1}{10}\ln\frac{47}{87}$$
$$T = 7 + 87e^{-\left(-\frac{1}{10}\ln\frac{47}{87}\right)t}$$
When $T = 18$:
$$18 = 7 + 87e^{-\left(-\frac{1}{10}\ln\frac{47}{87}\right)t}$$
$$e^{-\left(-\frac{1}{10}\ln\frac{47}{87}\right)t} = \frac{11}{87}$$
$$\left(\frac{1}{10}\ln\frac{47}{87}\right)t = \ln\frac{11}{87}$$

$$t = \frac{\ln \frac{11}{87}}{\left(\frac{1}{10}\ln \frac{47}{87}\right)} = 33.58 \text{ minutes past 6.00 pm}$$

So she can put the curry into the refrigerator at 6.34 pm.

9

$$\begin{aligned} \frac{dm}{dt} &= -km \\ \frac{1}{m}\frac{dm}{dt} &= -k \\ \int \frac{1}{m}\frac{dm}{dt}\,dt &= \int -k\,dt \\ \ln m &= -kt + c \\ m &= e^{-kt}e^{c} \\ m &= Ae^{-kt} \end{aligned}$$

Using the fact the $m = m_0$ when t $= 0$:

$$\begin{aligned} m_0 &= Ae^0 = A \\ A &= m_0 \\ m &= m_0 e^{-kt} \end{aligned}$$

Using the fact that $m = \frac{1}{2}m_0$ when $t = 5700$:

$$\begin{aligned} \frac{1}{2}m_0 &= m_0 e^{-5700k} \\ e^{-5700k} &= \frac{1}{2} \\ -5700k &= \ln \frac{1}{2} = -\ln 2 \\ k &= \frac{\ln 2}{5700} \\ m &= m_0 e^{-\left(\frac{\ln 2}{5700}\right)t} \end{aligned}$$

When $t = 2500$:

$$m = m_0 e^{-\left(\frac{\ln 2}{5700}\right)(2500)} = 0.74m_0$$

Approximately 74 % of the original amount would be present.

10 $\frac{dL}{dt} = k,\ k > 0$

$$\begin{aligned} \int \frac{dL}{dt} &= k dt = \int k\,dt \\ L &= kt + c \end{aligned}$$

Using the fact that $L = 20$ when $t = 0$:

$$\begin{aligned} 20 &= c \\ L &= kt + 20 \end{aligned}$$

Using the fact that $L = 26$ when $t = 20$:

$$\begin{aligned} 26 &= 20k + 20 \\ k &= \frac{6}{20} = \frac{3}{10} \\ L &= \frac{3}{10}t + 20 \end{aligned}$$

11 **a**

In Question **11** it is important to note that the temperature of the liquid starts below the temperature outside. Therefore, the difference between the temperature of the liquid and the temperature outside needs to be $24 - T$ so that it is positive.

$$\frac{dT}{dt} = k(24 - T)$$
$$\frac{1}{24 - T}\frac{dT}{dt} = k$$
$$\int \frac{1}{24 - T}\frac{dT}{dt}\,dt = \int k\,dt$$
$$-\ln|24 - T| = kt + c$$
$$24 - T = Ae^{-kt}$$
$$T = 24 - Ae^{-kt}$$

Using the fact that $T = 4$ when $t = 0$:

$$4 = 24 - A$$
$$A = 20$$
$$T = 24 - 20e^{-kt}$$

Using the fact that $T = 11$ when $t = 2$:

$$11 = 24 - 20e^{-2k}$$
$$e^{-2k} = \frac{13}{20}$$
$$-2k = \ln\frac{13}{20}$$
$$k = -\frac{1}{2}\ln\frac{13}{20} = \frac{1}{2}\ln\frac{20}{13}$$
$$T = 24 - 20e^{-\left(\frac{1}{2}\ln\frac{20}{13}\right)t}$$

When $T = 20$:

$$e^{-\left(\frac{1}{2}\ln\frac{20}{13}\right)t} = \frac{4}{20}$$
$$t = \frac{\ln\frac{4}{20}}{-\left(\frac{1}{2}\ln\frac{20}{13}\right)} = 7.47 \text{ minutes (to 3 s.f.)}$$

b $T = 24 - 20e^{-\left(\frac{1}{2}\ln\frac{20}{13}\right)t} \to 24 - 20(0) = 24^\circ\text{C}$

12 **a**

$$\frac{dn}{dt} = k\sqrt{n}, k > 0$$
$$n^{-\frac{1}{2}}\frac{dn}{dt} = k$$
$$\int n^{-\frac{1}{2}}\frac{dn}{dt}\,dt = \int k\,dt$$
$$2n^{\frac{1}{2}} = kt + c$$
$$2\sqrt{n} = kt + c$$

b Using the fact that $n = 0$ when $t = 0$:

$$0 = 0 + c$$
$$c = 0$$
$$2\sqrt{n} = kt$$

Using the fact that $n = 3600$ when $t = 6$:

$$2\sqrt{3600} = 6k$$
$$120 = 6k$$
$$k = 20$$
$$2\sqrt{n} = 20t$$

When $n = 6800$:

$$2\sqrt{6800} = 20t$$
$$t = \frac{\sqrt{6800}}{10}$$

9 complete months are required.

c When $n = 14\,000$:

$$2\sqrt{14000} = 20t$$
$$t = 11.8$$

The number of customers reaches 14 000 during the twelfth month.

13 **a**

$$\frac{dP}{dt} = kP$$
$$\frac{1}{P}\frac{dP}{dt} = k$$
$$\int \frac{1}{P}\frac{dP}{dt}\,dt = \int k\,dt$$
$$\ln P = kt + c$$
$$P = Ae^{kt}$$

Using the fact that $P = 5000$ when $t = 0$:

$$5000 = Ae^0 = A$$
$$P = 5000e^{kt}$$

Using the fact that $P = 1\,000\,000$ when $t = 3.5$:

$$1000000 = 5000e^{3.5k}$$
$$e^{3.5k} = 200$$
$$3.5k = \ln 200$$
$$k = \frac{1}{3.5}\ln 200$$
$$P = 5000e^{\left(\frac{1}{3.5}\ln 200\right)t}$$

When $P = 10\,000$:

$$10000 = 5000e^{\left(\frac{1}{3.5}\ln 200\right)t}$$
$$e^{\left(\frac{1}{3.5}\ln 200\right)t} = 2$$
$$\left(\frac{1}{3.5}\ln 200\right)t = \ln 2$$
$$t = \frac{\ln 2}{\frac{1}{3.5}\ln 200} = 0.458 \text{ (to 3 s.f.)}$$

b The model assumes that there is no limit to the number of bacteria, which is unrealistic.

END-OF-CHAPTER REVIEW EXERCISE 10

P3 **This exercise is for Pure Mathematics 3 students only.**

1

$$\frac{dy}{dx} = x^2 e^{y+2x}$$

$$e^{-y}\frac{dy}{dx} = x^2 e^{2x}$$

$$\int e^{-y}\frac{dy}{dx}\,dx = \int x^2 e^{2x}\,dx$$

Using integration by parts (twice):

$$-e^{-y} = \frac{1}{2}x^2e^{2x} - \int \frac{1}{2}e^{2x}(2x)\,dx$$

$$-e^{-y} = \frac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx$$

$$-e^{-y} = \frac{1}{2}x^2e^{2x} - \left\{\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx\right\}$$

$$-e^{-y} = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c$$

Using the fact that $y = 0$ when $x = 0$:

$$-1 = 0 + 0 + \frac{1}{4} + c$$

$$c = -\frac{5}{4}$$

$$-e^{-y} = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} - \frac{5}{4}$$

$$e^{-y} = \frac{5}{4} - \frac{1}{2}x^2e^{2x} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$$

$$y = -\ln\left(\frac{5}{4} - \frac{1}{2}x^2e^{2x} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right)$$

When using integration by parts twice, it is often best to use various types of bracket because you will normally need to embed pairs of brackets within other brackets. In worked solution 1, curly brackets are used to separate embedded calculations.

2 **a**

$$\frac{dx}{dt} = \frac{x(2500 - x)}{5000}$$

$$\frac{5000}{x(2500 - x)}\frac{dx}{dt} = 1$$

$$\int \frac{5000}{x(2500 - x)}\frac{dx}{dt}\,dt = \int 1\,dt$$

Using partial fractions:

$$\frac{5000}{x(2500 - x)} = \frac{A}{x} + \frac{B}{2500 - x}$$

$$5000 = A(2500 - x) + Bx$$

Letting $x = 0$:

$$5000 = 2500A$$

$$A = 2$$

Letting $x = 2500$:

$$5000 = 0 + 2500B$$

$$B = 2$$

$$\int \left(\frac{2}{x} + \frac{2}{2500 - x}\right)\frac{dx}{dt}\,dt = \int 1\,dt$$

$$2\ln|x| - 2\ln|2500 - x| = t + c$$

$$2\ln\left|\frac{x}{2500 - x}\right| = t + c$$

Using the fact that $x = 500$ when $t = 0$:

$$2\ln\frac{500}{2000}=c$$
$$c=2\ln 0.25$$
$$2\ln\left|\frac{x}{2500-x}\right|=t+2\ln\frac{1}{4}$$
$$2\ln\left|\frac{4x}{2500-x}\right|=t$$
$$\ln\left|\frac{4x}{2500-x}\right|=\frac{t}{2}$$
$$\frac{4x}{2500-x}=e^{\frac{t}{2}}$$
$$4x=2500e^{\frac{t}{2}}-xe^{\frac{t}{2}}$$
$$(4+e^{\frac{t}{2}})x=2500e^{\frac{t}{2}}$$
$$x=\frac{2500e^{\frac{t}{2}}}{4+e^{\frac{t}{2}}}$$

b $x=\frac{2500e^{\frac{t}{2}}}{4+e^{\frac{t}{2}}}\to\frac{2500}{1}=2500$

Read the $\to$ symbol used as 'tends to'.

3
$$y^3\frac{dy}{dx}=1+y^4$$
$$\frac{y^3}{1+y^4}\frac{dy}{dx}=1$$
$$\int\frac{y^3}{1+y^4}\frac{dy}{dx}\,dx=\int 1\,dx$$
$$\frac{1}{4}\ln\left(1+y^4\right)=x+c$$
$$\ln\left(1+y^4\right)=4x+B$$
$$1+y^4=e^{4x+B}=Ce^{4x}$$

Using the fact that $y=2$ when $x=0$:

$$1+16=Ce^0=C$$
$$C=17$$
$$y^4=17e^{4x}-1$$

4 **a** $\frac{dy}{dx}=kx\sqrt{y}$

Using the fact that (3, 4) the gradient is – 5:

$$-5=3k\sqrt{4}=6k$$
$$k=-\frac{5}{6}$$
$$y^{-\frac{1}{2}}\frac{dy}{dx}=-\frac{5}{6}x$$
$$\int y^{-\frac{1}{2}}\frac{dy}{dx}\,dx=\int-\frac{5}{6}x\,dx$$
$$2y^{\frac{1}{2}}=-\frac{5}{12}x^2+c$$

Using the fact that $y=4$ when $x=3$:

$$4=-\frac{45}{12}+c$$
$$c=\frac{93}{12}$$
$$2y^{\frac{1}{2}}=-\frac{5}{12}x^2+\frac{93}{12}$$
$$24\sqrt{y}=93-5x^2$$

b $\frac{dy}{dx} = -\frac{5}{6}x\sqrt{y}$

At $(-3, 4)$:

$$\frac{dy}{dx} = -\frac{5}{6}(-3)\sqrt{4}$$
$$= 5$$

5 **a**

You need to use partial fractions first.

$$\frac{50}{(5-x)(10-x)} = \frac{A}{5-x} + \frac{B}{10-x}$$
$$50 = A(10-x) + B(5-x)$$

Letting $x = 10$:

$$50 = 0 - 5B$$
$$B = -10$$

Letting $x = 5$:

$$50 = 5A$$
$$A = 10$$

$$\frac{50}{(5-x)(10-x)} = \frac{10}{5-x} - \frac{10}{10-x}$$
$$\int \frac{50}{(5-x)(10-x)}\,dx = \int \left(\frac{10}{5-x} - \frac{10}{10-x}\right) dx$$
$$= -10\ln(5-x) + 10\ln(10-x) + c$$

Note that you do not need to use the modulus function because $x < 5$ and both $5 - x$ and $10 - x$ are positive.

b **i** $\frac{dx}{dt} = k(5-x)(10-x)$

Using the fact that $\frac{dx}{dt} = 1$ when $x = 0$:

$$1 = 50k$$
$$k = \frac{1}{50}$$
$$\frac{dx}{dt} = \frac{1}{50}(5-x)(10-x)$$
$$50\frac{dx}{dt} = (5-x)(10-x)$$

ii

$$\frac{50}{(5-x)(10-x)}\frac{dx}{dt} = 1$$
$$\int \frac{50}{(5-x)(10-x)}\frac{dx}{dt}\,dt = \int 1\,dt$$
$$-10\ln(5-x) + 10\ln(10-x) = t + c$$
$$10\ln\left(\frac{10-x}{5-x}\right) = t + c$$

Using the fact that $x = 0$ when $t = 0$:

$$10\ln 2 = c$$
$$10\ln\left(\frac{10-x}{5-x}\right) = t + 10\ln 2$$
$$10\ln\left(\frac{10-x}{2(5-x)}\right) = t$$
$$\frac{10-x}{2(5-x)} = \mathrm{e}^{\frac{t}{10}}$$
$$10 - x = 10\mathrm{e}^{\frac{t}{10}} - 2x\mathrm{e}^{\frac{t}{10}}$$
$$x\left(2\mathrm{e}^{\frac{t}{10}} - 1\right) = 10\mathrm{e}^{\frac{t}{10}} - 10$$
$$x = \frac{10\left(\mathrm{e}^{\frac{t}{10}} - 1\right)}{2\mathrm{e}^{\frac{t}{10}} - 1}$$

iii $x = \dfrac{10\left(\mathrm{e}^{\frac{t}{10}} - 1\right)}{2\mathrm{e}^{\frac{t}{10}} - 1} \to \dfrac{10}{2} = 5$ grams as $t \to \infty$

6
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x\mathrm{e}^x}{5y^4}$$
$$5y^4\frac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^x$$
$$\int 5y^4\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int x\mathrm{e}^x\,\mathrm{d}x$$

Using integration by parts on the right-hand integral:

$$y^5 = \int x\mathrm{e}^x\,\mathrm{d}x = x\mathrm{e}^x - \int \mathrm{e}^x\,\mathrm{d}x$$
$$y^5 = x\mathrm{e}^x - \mathrm{e}^x + c$$

Using the fact that $y = 4$ when $x = 0$:

$$1024 = 0 - 1 + c$$
$$c = 1025$$
$$y^5 = x\mathrm{e}^x - \mathrm{e}^x + 1025$$

When $x = 3.5$:

$y = \sqrt[5]{(3.5\mathrm{e}^{3.5} - \mathrm{e}^{3.5} + 1025)} = 4.06$ (to 3 significant figures)

7 a
$$u = \sqrt{y^4 - 1} = \left(y^4 - 1\right)^{\frac{1}{2}}$$
$$\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{2}\left(y^4 - 1\right)^{-\frac{1}{2}}\left(4y^3\right)$$
$$= \frac{2y^3}{\sqrt{y^4 - 1}}$$
$$\int \frac{y^3}{\sqrt{y^4 - 1}}\,\mathrm{d}y = \int \frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}y}\,\mathrm{d}y$$
$$= \int \frac{1}{2}\,\mathrm{d}u$$
$$= \frac{1}{2}u + c$$
$$= \frac{1}{2}\sqrt{y^4 - 1} + c$$

b
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(2x+1)\sqrt{y^4 - 1}}{xy^3}$$
$$\frac{y^3}{\sqrt{y^4 - 1}}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x+1}{x}$$
$$\int \frac{y^3}{\sqrt{y^4 - 1}}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int\left(2 + \frac{1}{x}\right)\mathrm{d}x$$
$$\frac{1}{2}\sqrt{y^4 - 1} = 2x + \ln|x| + c$$

Using the fact that $y = \sqrt{3}$ when $x = 1$:

$$\frac{1}{2}\sqrt{9-1} = 2 + \ln 1 + c$$
$$c = \sqrt{2} - 2$$
$$\frac{1}{2}\sqrt{y^4 - 1} = 2x + \ln|x| + \sqrt{2} - 2$$

8 **a** $\frac{\mathrm{d}h}{\mathrm{d}t} = k(8-h),\, k > 0$

$$\frac{1}{8-h}\frac{\mathrm{d}h}{\mathrm{d}t} = k$$
$$\int \frac{1}{8-h}\frac{\mathrm{d}h}{\mathrm{d}t}\,\mathrm{d}t = \int k\,\mathrm{d}t$$
$$-\ln|8-h| = kt + c$$
$$\ln|8-h| = -kt + c'$$
$$8 - h = A\mathrm{e}^{-kt}$$
$$h = 8 - A\mathrm{e}^{-kt}$$

Using the fact that $h = 0.5$ when $t = 0$:

$$0.5 = 8 - A\mathrm{e}^{0}$$
$$A = 7.5$$
$$h = 8 - 7.5\mathrm{e}^{-kt}$$

Using the fact that $h = 2$ when $t = 5$:

$$2 = 8 - 7.5\mathrm{e}^{-5k}$$
$$\mathrm{e}^{-5k} = \frac{6}{7.5} = 0.8$$
$$-5k = \ln 0.8$$
$$k = -\frac{1}{5}\ln 0.8$$
$$h = 8 - 7.5\mathrm{e}^{\left(\frac{1}{5}\ln 0.8\right)t}$$

Note that when multiplying an equation containing an arbitrary constant, you should show that the constant would be modified too. In worked solution **a**, c' is used to show that c has changed to a different arbitrary constant.

b Note that $\ln 0.8$ is negative. Indeed, $\ln x$ is negative if $x < 1$.

$\mathrm{e}^{\left(\frac{1}{5}\ln 0.8\right)t} \to 0$ as $t \to \infty$

$h \to 8$ metres

9

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{3(x+y)}$$
$$\frac{1}{\mathrm{e}^{3y}}\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{3x}$$
$$\mathrm{e}^{-3y}\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{3x}$$
$$\int \mathrm{e}^{-3y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \mathrm{e}^{3x}\,\mathrm{d}x$$
$$-\frac{1}{3}\mathrm{e}^{-3y} = \frac{1}{3}\mathrm{e}^{3x} + c$$
$$\mathrm{e}^{-3y} = -\mathrm{e}^{3x} + c'$$

Using the fact that $y = 0$ when $x = 0$:

$$1 = -1 + c'$$
$$c' = 2$$
$$-3y = \ln(2 - \mathrm{e}^{3x})$$
$$y = -\frac{1}{3}\ln(2 - \mathrm{e}^{3x})$$

10 **a** $\frac{\mathrm{d}x}{\mathrm{d}t} = -kxt$

b $$\frac{1}{x}\frac{dx}{dt} = -kt$$
$$\int \frac{1}{x}\frac{dx}{dt}\,dt = \int -kt\,dt$$
$$\ln x = \frac{-kt^2}{2} + c$$

c Using the result from part **b**:

$$x = Ae^{\frac{-kt^2}{2}}$$

Using the fact that $x = 150$ when $t = 0$:

$$150 = Ae^0$$
$$x = 150e^{\frac{-kt^2}{2}}$$

Using the fact that $x = 120$ when $t = 10$:

$$120 = 150e^{-50k}$$
$$e^{-50k} = 0.8$$
$$k = -\frac{1}{50}\ln 0.8$$
$$x = 150e^{\left(\frac{1}{100}\ln 0.8\right)t^2}$$

When $x = 1$:

$$e^{\left(\frac{1}{100}\ln 0.8\right)t^2} = \frac{1}{150}$$
$$\left(\frac{1}{100}\ln 0.8\right)t^2 = \ln\frac{1}{150}$$

$t = 47.4\,\text{s}$ (to 3 significant figures)

11 **a**

> If you notice that you are going to need partial fractions, then it is best to work them out at the beginning, so that they don't interrupt the actual solution.

$$\frac{1}{P(5-P)} = \frac{A}{P} + \frac{B}{5-P}$$
$$1 = A(5-P) + BP$$

Letting $P = 0$:

$$1 = 5A$$
$$A = \frac{1}{5}$$

Letting $P = 5$:

$$1 = 0 + 5B$$
$$B = \frac{1}{5}$$

$$\frac{1}{P(5-P)} = \frac{1}{5P} + \frac{1}{5\,(5-P)}$$
$$\int \frac{1}{P(5-P)}\,dP = \int \left(\frac{1}{5P} + \frac{1}{5\,(5-P)}\right)dP$$
$$= \frac{1}{5}\ln|P| - \frac{1}{5}\ln|5-P| + c$$

b $$\frac{dP}{dt} = P(5-P)$$
$$\frac{1}{P(5-P)}\frac{dP}{dt} = 1$$
$$\int \frac{1}{P(5-P)}\frac{dP}{dt}\,dt = \int 1\,dt$$
$$\frac{1}{5}\ln|P| - \frac{1}{5}\ln|5-P| = t + c$$

Using the fact that $P = 3$ when $t = 0$:

$$\frac{1}{5}\ln|3| - \frac{1}{5}\ln|2| = c$$

$$\frac{1}{5}\ln|P| - \frac{1}{5}\ln|5-P| = t + \frac{1}{5}\ln\frac{3}{2}$$

$$\frac{1}{5}\ln\left(\frac{2P}{3(5-P)}\right) = t$$

$$\ln\left(\frac{2P}{3(5-P)}\right) = 5t$$

$$\frac{2P}{3(5-P)} = e^{5t}$$

$$2P = (15-3P)e^{5t}$$

$$2P = 15e^{5t} - 3Pe^{5t}$$

$$P(2+3e^{5t}) = 15e^{5t}$$

$$P = \frac{15e^{5t}}{2+3e^{5t}}$$

c $P = \dfrac{15e^{5t}}{2+3e^{5t}} \to \dfrac{15}{3} = 5$ as $t \to \infty$

12

> Note that P is the number of millions. So a population of 50 million means that $P = 50$.

$$\frac{100}{P(100-P)}\frac{dP}{dt} = k$$

$$\frac{100}{P(100-P)} = \frac{A}{P} + \frac{B}{100-P}$$

$$100 = A(100-P) + BP$$

Letting $P = 0$:

$$100 = 100A$$
$$A = 1$$

Letting $P = 100$:

$$100 = 100B$$
$$B = 1$$

$$\left(\frac{1}{P} + \frac{1}{100-P}\right)\frac{dP}{dt} = k$$

$$\int\left(\frac{1}{P} + \frac{1}{100-P}\right)\frac{dP}{dt}\,dt = \int k\,dt$$

$$\ln|P| - \ln|100-P| = kt + c$$

Using the fact that $P = 50$ when $t = 0$:

$$\ln 50 - \ln 50 = 0 + c$$
$$c = 0$$

$$\ln|P| - \ln|100-P| = kt$$

Using the fact that $P = 60$ when $t = 10$:

$$\ln 60 - \ln 40 = 10k$$
$$k = \frac{1}{10}\ln 1.5$$

$$\ln|P| - \ln|100-P| = \left(\frac{1}{10}\ln 1.5\right)t$$

$$\ln\left(\frac{P}{100-P}\right) = \left(\frac{1}{10}\ln 1.5\right)t$$

$$\frac{P}{100-P} = e^{\left(\frac{1}{10}\ln 1.5\right)t}$$

$$P = 100e^{\left(\frac{1}{10}\ln 1.5\right)t} - Pe^{\left(\frac{1}{10}\ln 1.5\right)t}$$

$$P\left(1 + e^{\left(\frac{1}{10}\ln 1.5\right)t}\right) = 100e^{\left(\frac{1}{10}\ln 1.5\right)t}$$

$$P = \frac{100e^{\left(\frac{1}{10}\ln 1.5\right)t}}{1 + e^{\left(\frac{1}{10}\ln 1.5\right)t}}$$

When $t = 25$:

$P = 73.4$ million (to 3 significant figures)

13 **a**

$$u = x^2$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$
$$x\,\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$$

$$\int x \sin x^2\,\mathrm{d}x$$
$$= \int \frac{1}{2}\sin u\,\mathrm{d}u$$
$$= -\frac{1}{2}\cos u + c$$
$$= -\frac{1}{2}\cos x^2 + c$$

b

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x \sin x^2}{y}$$
$$y\frac{\mathrm{d}y}{\mathrm{d}x} = x \sin x^2$$
$$\int y\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int x \sin x^2\,\mathrm{d}x$$
$$\frac{1}{2}y^2 = -\frac{1}{2}\cos x^2 + c$$

Using the fact that $y = -1$ when $x = 0$:

$$\frac{1}{2} = -\frac{1}{2} + c$$
$$c = 1$$
$$\frac{1}{2}y^2 = -\frac{1}{2}\cos x^2 + 1$$

14

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\tan x}{\mathrm{e}^{3y}}$$
$$\mathrm{e}^{3y}\frac{\mathrm{d}y}{\mathrm{d}x} = \tan x$$
$$\int \mathrm{e}^{3y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \tan x\,\mathrm{d}x$$
$$\frac{1}{3}\mathrm{e}^{3y} = \ln|\sec x| + c$$

Using the fact that $y = 0$ when $x = 0$:

$$\frac{1}{3} = \ln 1 + c$$
$$c = \frac{1}{3}$$
$$\frac{1}{3}\mathrm{e}^{3y} = \ln|\sec x| + \frac{1}{3}$$
$$\mathrm{e}^{3y} = 3\,|\sec x| + 1$$
$$y = \frac{1}{3}\ln(3\,|\sec x| + 1)$$
$$= \frac{1}{3}\ln(1 - 3\ln|\cos x|)$$

Remember that

$$\int \tan x\,\mathrm{d}x$$
$$= \int \frac{\sin x}{\cos x}\,\mathrm{d}x$$
$$= -\int \frac{-\sin x}{\cos x}\,\mathrm{d}x$$
$$= -\ln|\cos x| + c$$
$$= \ln|\sec x| + c$$

15 **a** $\frac{\mathrm{d}x}{\mathrm{d}t} = kx(2000 - x)$

Using the fact that $\frac{\mathrm{d}x}{\mathrm{d}t} = 50$ when $x = 500$:

$$50 = 500k(1500)$$
$$k = \frac{1}{15000}$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{15000}x(2000 - x)$$
$$\frac{15000}{x(2000 - x)}\frac{\mathrm{d}x}{\mathrm{d}t} = 1$$
$$\frac{15000}{x(2000 - x)} = \frac{A}{x} + \frac{B}{2000 - x}$$
$$15000 = A(2000 - x) + Bx$$

Letting $x = 0$:

$$15000 = 2000A$$
$$A = \frac{15}{2}$$

Letting $x = 2000$:

$$15000 = 2000B$$
$$B = \frac{15}{2}$$
$$\left(\frac{15}{2x} + \frac{15}{2\,(2000 - x)}\right)\frac{\mathrm{d}x}{\mathrm{d}t} = 1$$
$$\frac{15}{2}\int\left(\frac{1}{x} + \frac{1}{2000 - x}\right)\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int 1\,\mathrm{d}t$$
$$\frac{15}{2}(\ln|x| - \ln|2000 - x|) = t + c$$

Using the fact that $x = 500$ when $t = 0$:

$$\frac{15}{2}(\ln 500 - \ln 1500) = c$$
$$c = \frac{15}{2}\ln\frac{1}{3}$$
$$\frac{15}{2}(\ln|x| - \ln|2000 - x|) = t + \frac{15}{2}\ln\frac{1}{3}$$
$$t = \frac{15}{2}\left(\ln|x| - \ln|2000 - x| - \ln\frac{1}{3}\right)$$
$$t = \frac{15}{2}\ln\left(\frac{3x}{2000 - x}\right)$$

b When $x = 1900$:

$$t = \frac{15}{2}\ln\left(\frac{3(1900)}{2000 - 1900}\right) = 30.3 \text{ hours (to 3 significant figures)}$$

16

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - y^2$$
$$\frac{1}{(1 - y)(1 + y)}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$
$$\frac{1}{(1 - y)(1 + y)} = \frac{A}{1 - y} + \frac{B}{1 + y}$$
$$1 = A(1 + y) + B(1 - y)$$

Letting $y = 1$:

$$1 = 2A$$
$$A = \frac{1}{2}$$

Letting $y = -1$:

$$1 = 2B$$
$$B = \frac{1}{2}$$
$$\frac{1}{(1-y)(1+y)} = \frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right)$$
$$\frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$
$$\int \frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x$$
$$\frac{1}{2}(-\ln|1-y| + \ln|1+y|) = \ln|x| + c$$
$$\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = \ln|x| + c$$

Using the fact that $y = 0$ when $x = 2$:

$$-\frac{1}{2}\ln 1 = \ln 2 + c$$
$$c = -\ln 2$$
$$\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = \ln|x| - \ln 2$$
$$\ln\sqrt{\frac{1+y}{1-y}} = \ln\frac{x}{2}$$
$$\sqrt{\frac{1+y}{1-y}} = \frac{x}{2}$$
$$\frac{1+y}{1-y} = \frac{x^2}{4}$$
$$4 + 4y = x^2 - x^2y$$
$$y\left(4 + x^2\right) = x^2 - 4$$
$$y = \frac{x^2-4}{x^2+4}$$

17

For Question **17** you will need to use the fact that $\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$.

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = (x+2)\sin^2 2\theta$$
$$\frac{1}{x+2}\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sin^2 2\theta$$
$$\int \frac{1}{x+2}\frac{\mathrm{d}x}{\mathrm{d}\theta}\,\mathrm{d}\theta = \int \sin^2 2\theta\,\mathrm{d}\theta$$
$$\ln|x+2| = \int\left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right)\mathrm{d}\theta$$
$$\ln|x+2| = \frac{1}{2}\theta - \frac{1}{8}\sin 4\theta + c$$

Using the fact that $x = 0$ when $\theta = 0$:

$$\ln 2 = 0 - 0 + c$$
$$\ln|x+2| = \frac{1}{2}\theta - \frac{1}{8}\sin 4\theta + \ln 2$$

When $\theta = \frac{\pi}{4}$:

$$\ln|x+2| = \frac{\pi}{8} - 0 + \ln 2$$
$$x = \mathrm{e}^{\frac{\pi}{8} + \ln 2} - 2 = 0.961$$

Chapter 11
Complex numbers

P3 This chapter is for Pure Mathematics 3 students only.

EXERCISE 11A

1 **a** $\sqrt{-144} = \sqrt{144 \times -1}$

$= \sqrt{144} \times \sqrt{-1}$

$= 12\text{i}$

d $\sqrt{-16} + \sqrt{-81}$

$= \sqrt{16}\sqrt{-1} + \sqrt{81}\sqrt{-1}$

$= 4\text{i} + 9\text{i}$

$= 13\text{i}$

2 **b** $9\text{i} - \left(\text{i}\sqrt{2}\right)^3$

$= 9\text{i} - \text{i}^3\left(\sqrt{2}\right)^3$

$= 9\text{i} - \text{i}^2\text{i}\left(\sqrt{2}\right)^2\sqrt{2}$

$= 9\text{i} - (-1)\text{i}(2\sqrt{2})$

$= 9\text{i} + 2\text{i}\sqrt{2}$

$= \left(9 + 2\sqrt{2}\right)\text{i}$

> Remember that
>
> $\text{i}^2 = -1$
>
> $\text{i}^3 = \text{i}^2\text{i} = -\text{i}$
>
> $\text{i}^4 = \left(\text{i}^2\right)^2 = (-1)^2 = 1$
>
> When simplifying powers of i greater than 4, use these facts to help you simplify.
>
> For example, $\text{i}^7 = \text{i}^4\text{i}^2\text{i} = 1(-1)\text{i} = -\text{i}$.

d $-\dfrac{5}{6\text{i}^2}$

$= -\dfrac{5}{6(-1)}$

$= \dfrac{5}{6}$

3 **a** $x^2 + \dfrac{64}{25} = 0$

$x^2 = -\dfrac{64}{25}$

$x = \pm\sqrt{-\dfrac{64}{25}}$

$= \pm\sqrt{\dfrac{64}{25}}\sqrt{-1}$

$= \pm\dfrac{8}{5}\text{i}$

b $4x^2 + 7 = 0$

$$x^2 = -\frac{7}{4}$$

$$x = \pm\sqrt{\frac{7}{4}}\sqrt{-1}$$

$$= \pm\frac{\sqrt{7}}{2}\mathrm{i}$$

c $12x^2 + 3 = 0$

$$x^2 = -\frac{1}{4}$$

$$x = \pm\sqrt{-\frac{1}{4}}$$

$$= \pm\sqrt{\frac{1}{4}}\sqrt{-1}$$

$$\pm\frac{1}{2}\mathrm{i}$$

EXERCISE 11B

1 **b** $z_1^* = 5 + 3\text{i}$

$z_1^* - z_2$

$= 5 + 3\text{i} - (1 + 2\text{i})$

$= 4 + \text{i}$

d $$\frac{z_1}{z_2} = \frac{5 - 3\text{i}}{1 + 2\text{i}}$$

$$= \frac{(5 - 3\text{i})(1 - 2\text{i})}{(1 + 2\text{i})(1 - 2\text{i})}$$

$$= \frac{5 - 10\text{i} - 3\text{i} + 6\text{i}^2}{1 - 4\text{i}^2}$$

$$= \frac{5 + 6(-1) - 13\text{i}}{1 - 4(-1)}$$

$$= \frac{-1 - 13\text{i}}{5}$$

$$= -\frac{1}{5} - \frac{13}{5}\text{i}$$

2

> Two methods are shown here: worked solution **b** uses completing the square and worked solution **e** uses the quadratic formula. These methods are, in fact, the same thing because the quadratic formula is derived using completing the square.

b $$z^2 + 4z + 5 = 0$$

$$(z + 2)^2 - 4 + 5 = 0$$

$$(z + 2)^2 = -1$$

$$z + 2 = \pm\text{i}$$

$$z = -2 \pm \text{i}$$

e $3z^2 + 8z + 10 = 0$

$$z = \frac{-8 \pm \sqrt{8^2 - 4(3)(10)}}{2(3)}$$

$$= \frac{-8 \pm \sqrt{64 - 120}}{6}$$

$$= \frac{-8 \pm \sqrt{-56}}{6}$$

$$= \frac{-8 \pm \text{i}\sqrt{4}\sqrt{14}}{6}$$

$$= \frac{-8 \pm 2\text{i}\sqrt{14}}{6}$$

$$= -\frac{4}{3} \pm \frac{\sqrt{14}}{3}\text{i}$$

3 **a** $(x + 2y) + \text{i}(3x - y) = 1 + 10\text{i}$

Equating real parts:

$x + 2y = 1$ ....[1]

Equating imaginary parts:

$3x - y = 10$

$6x - 2y = 20$ ....[2]

[1] +[2]:

$7x = 21$

$x = 3$

$3 + 2y = 1$

$2y = -2$

$y = -1$

b $(x+y-4)+2x\text{i}=(5-y)\text{i}$

Equating real parts:

$x+y-4=0$

$x+y=4$ ....[1]

Equating imaginary parts:

$2x=5-y$

$2x+y=5$ ....[2]

[2] −[1]:

$x=1$

$1+y=4$

$y=3$

c $(x-y)+(2x-y)\text{i}=-1$

Equating real parts:

$x-y=-1$ ....[1]

Equating imaginary parts:

$2x-y=0$ ....[2]

[2] − [1]:

$x=1$

$1-y=-1$

$y=2$

4 **c** $(7-3\text{i})^2$

$=(7-3\text{i})(7-3\text{i})$

$=49-21\text{i}-21\text{i}+9\text{i}^2$

$=49+9(-1)-42\text{i}$

$=40-42\text{i}$

g $\dfrac{13(1+\text{i})}{2+3\text{i}}$

$=\dfrac{13(1+\text{i})(2-3\text{i})}{(2+3\text{i})(2-3\text{i})}$

$=\dfrac{13(2-3\text{i}+2\text{i}-3\text{i}^2)}{4-9\text{i}^2}$

$=\dfrac{26-39\text{i}+26\text{i}+39}{4+9}$

$=\dfrac{65-13\text{i}}{13}$

$=5-\text{i}$

5 **a** $\left(\dfrac{1}{5}+\dfrac{2\sqrt{6}}{5}\text{i}\right)^2$

$=\dfrac{1}{25}+\dfrac{4\sqrt{6}}{25}\text{i}+\dfrac{4\times 6}{25}\text{i}^2$

$=\dfrac{1}{25}+\dfrac{4\sqrt{6}}{25}\text{i}-\dfrac{24}{25}$

$=-\dfrac{23}{25}+\dfrac{4\sqrt{6}}{25}\text{i}$

$5\left(\dfrac{1}{5}+\dfrac{2\sqrt{6}}{5}\text{i}\right)^2-2\left(\dfrac{1}{5}+\dfrac{2\sqrt{6}}{5}\text{i}\right)+5$

$=5\left(-\dfrac{23}{25}+\dfrac{4\sqrt{6}}{25}\text{i}\right)-2\left(\dfrac{1}{5}+\dfrac{2\sqrt{6}}{5}\text{i}\right)+5$

$=-\dfrac{23}{5}+\dfrac{4\sqrt{6}}{5}\text{i}-\dfrac{2}{5}-\dfrac{4\sqrt{6}}{5}\text{i}+5$

$=-5+5+\dfrac{4\sqrt{6}}{5}-\dfrac{4\sqrt{6}}{5}$

$=0$

Hence $\frac{1}{5}+\frac{2\sqrt{6}}{5}\text{i}$ is a solution to the equation $5z^2-2z+5=0$

b Remember that if z is the root of a quadratic equation then z^* is also a root.

The other solution is the complex conjugate, $\frac{1}{5}-\frac{2\sqrt{6}}{5}\text{i}$

6 **b** $\alpha+\beta=1+5\text{i}+1-5\text{i}$

$=2$

$\alpha\beta=(1+5\text{i})(1-5\text{i})$

$=1-25\text{i}^2$

$=26$

$z^2-2z+26=0$

The tip given in the coursebook has been used here.

d $\alpha+\beta$

$=-5$

$\alpha\beta=\left(-\frac{5}{2}-\frac{\sqrt{31}}{2}\text{i}\right)\left(-\frac{5}{2}+\frac{\sqrt{31}}{2}\text{i}\right)$

$=\left(\frac{5}{2}\right)^2-\left(\frac{\sqrt{31}}{2}\right)^2\text{i}^2$

$=\frac{25}{4}+\frac{31}{4}$

$=\frac{56}{4}$

$=14$

$z^2+5z+14=0$

7 $z+4=3\text{i}z^*$

Let $z=x+\text{i}y$

$(x+\text{i}y)+4=3\text{i}(x-\text{i}y)$

Equating real parts:

$x+4=3y$

$x-3y=-4$ ....[1]

Equating imaginary parts:

$y=3x$ ....[2]

Substituting [2] into [1]:

$x-3(3x)=-4$

$x-9x=-4$

$-8x=-4$

$x=\frac{1}{2}$

$y=\frac{3}{2}$

8 $u(3-5\mathrm{i}) = 13+\mathrm{i}$

$$u = \frac{13+\mathrm{i}}{3-5\mathrm{i}}$$
$$= \frac{(13+\mathrm{i})(3+5\mathrm{i})}{(3-5\mathrm{i})(3+5\mathrm{i})}$$
$$= \frac{39+65\mathrm{i}+3\mathrm{i}+5\mathrm{i}^2}{9-25\mathrm{i}^2}$$
$$= \frac{39-5+68\mathrm{i}}{9+25}$$
$$= \frac{34+68\mathrm{i}}{34}$$
$$= 1+2\mathrm{i}$$

9 $5-\mathrm{i}\sqrt{3}$ is also a root of the equation.

$\alpha = 5+\mathrm{i}\sqrt{3}$

$\beta = 5-\mathrm{i}\sqrt{3}$

$\alpha+\beta = 10$

$$\alpha\beta = 25-\mathrm{i}^2\left(\sqrt{3}\right)^2$$
$$= 25+3 = 28$$

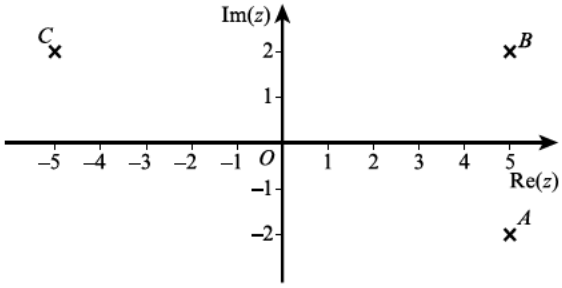
$z^2-10z+28=0$

10 $240 = \text{current} \times (48+36\mathrm{i})$

$$\text{current} = \frac{240}{48+36\mathrm{i}}$$
$$= \frac{240(48-36\mathrm{i})}{(48+36\mathrm{i})(48-36\mathrm{i})}$$
$$= \frac{240(48-36\mathrm{i})}{48^2+36^2}$$
$$= \frac{240(48-36\mathrm{i})}{3600}$$
$$= \frac{48-36\mathrm{i}}{15}$$
$$= \frac{16}{5}-\frac{12}{5}\mathrm{i}$$
$$= 3.2 - 2.4i \text{ amperes}$$

EXERCISE 11C

1 **a**



b $u = 5 - 2\text{i}$

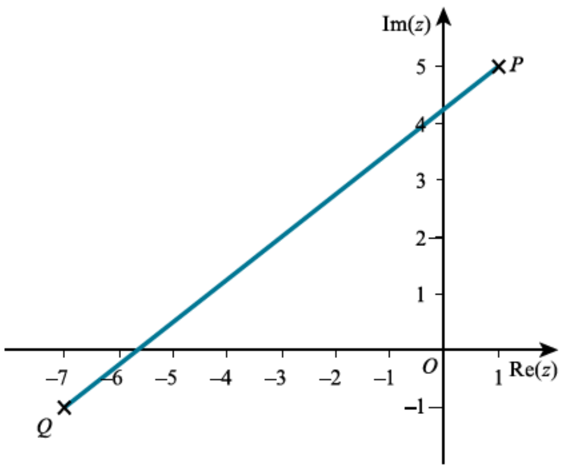
$u^* = 5 + 2\text{i}$

$-u = -5 + 2\text{i}$

As soon as you draw the Argand diagram it is immediately clear that the point you need has co-ordinates $(-5,\ -2)$.

So the complex number is $-5 - 2\text{i}$.

2 **a**



b You can see from the Argand diagram that the co-ordinates of the midpoint will be $(-3,\ 2)$. So the complex number will be $-3 + 2\text{i}$.

Alternatively, you can take the means of real and imaginary parts to get:

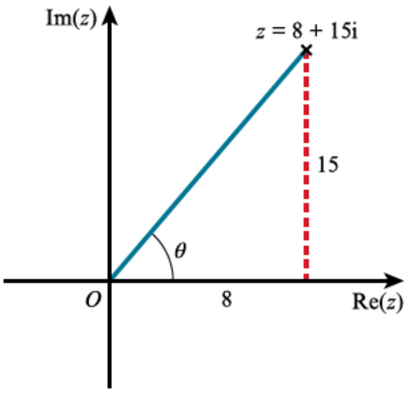
$$\frac{1+(-7)}{2} + \text{i}\left(\frac{5+(-1)}{2}\right)$$

$$= -\frac{6}{2} + \text{i}\left(\frac{4}{2}\right)$$

$$= -3 + 2\text{i}$$

Remember that the midpoint of two points A and B, with coordinates $(x_1,\ y_1)$ and $(x_2,\ y_2)$is $\left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}\right)$. This is the equivalent to taking the means of real parts and imaginary parts in complex numbers.

3 **c**

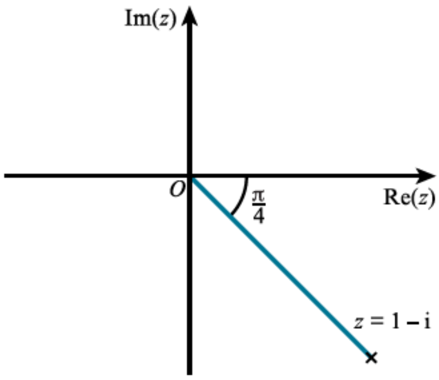


$$\text{Arg}(8+15\text{i}) = \theta = \tan^{-1}\left(\frac{15}{8}\right) = 1.08$$

$$|8+15\text{i}| = \sqrt{8^2+15^2} = \sqrt{289} = 17$$

Draw an Argand diagram to help you with your calculations such as finding arguments, working out angles and solving geometry problems.

i First consider the complex number $1 - \text{i}$:



Any positive multiple of a complex number has the same argument.

$$\text{Arg}\left[k(1-\text{i})\right] = -\frac{\pi}{4}$$

$$|k(1-\text{i})| = k\,|1-\text{i}| = k\sqrt{1^2+(-1)^2} = k\sqrt{2}$$

4 **a** Point A:

$$z_1 = -1 + 3\text{i}$$

$$|z_1| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\arg z = \pi - \tan^{-1} 3 = 1.89$$

$$-1 + 3\text{i} = \sqrt{10}(\cos(1.89) + \text{i}\sin(1.89))$$

Point B:

$$z_2 = 3 + i$$

$$|z_2| = \sqrt{3^2+1^2} = \sqrt{10}$$

$$\arg z = \tan^{-1}\left(\frac{1}{3}\right) = 0.322$$

$$3 + i = \sqrt{10}(\cos(0.322) + i\sin(0.322))$$

Point C:

$$z_3 = 1 - 3\text{i}$$

$$|z_3| = \sqrt{1 + (-3)^2} = \sqrt{10}$$

$$\arg z = -\tan^{-1} 3 = -1.25$$

$$1 - 3\text{i} = \sqrt{10}(\cos(-1.25) + \text{i}\sin(-1.25))$$

b $AB = |3 + \text{i} - (-1 + 3\text{i})|$
$= |4 - 2\text{i}|$
$= \sqrt{16 + 4}$
$= \sqrt{20}$
$AC = |1 - 3\text{i} - (-1 + 3\text{i})|$
$= |2 - 6\text{i}|$
$= \sqrt{4 + 36}$
$= \sqrt{40}$
$BC = |1 - 3\text{i} - (3 + \text{i})|$
$= |-2 - 4\text{i}|$
$= \sqrt{4 + 16}$
$= \sqrt{20}$

Note that $BC^2 + AB^2 = 20 + 20 = 40 = AC^2$ and $AB = BC$ so the triangle is right-angled and isosceles.

5 **a** $3\left(\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}\right)$
$= 3\left(\frac{1}{2} + \text{i}\frac{\sqrt{3}}{2}\right)$
$= \frac{3}{2} + \frac{3\sqrt{3}}{2}\text{i}$

d $3e^{-\frac{\text{i}\pi}{4}}$
$= 3\left(\cos\left(-\frac{\pi}{4}\right) + \text{i}\sin\left(-\frac{\pi}{4}\right)\right)$
$= 3\frac{\sqrt{2}}{2} - 3\frac{\sqrt{2}}{2}\text{i}$
$= \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}\text{i}$

6

Remember that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

$$z(4 - 9\text{i}) = 8 + 3\text{i}$$
$$z = \frac{8 + 3\text{i}}{4 - 9\text{i}}$$
$$|z^2| = |z|^2 = \frac{|8 + 3\text{i}|^2}{|4 - 9\text{i}|^2} = \frac{8^2 + 3^2}{4^2 + (-9)^2} = \frac{73}{97}$$
$$\arg z^2 = 2\arg z = 2\left(\arg(8 + 3\text{i}) - \arg(4 - 9\text{i})\right) = 2\tan^{-1}\left(\frac{3}{8}\right) - 2\tan^{-1}\left(-\frac{9}{4}\right) = 3.02$$

Remember that

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

7 **a** $|w| = 5$
$\arg w = \frac{\pi}{6}$
$w = 5\text{e}^{\text{i}\frac{\pi}{6}}$

b $$|z| = \frac{|3 - 7\text{i}|}{|5 - 2\text{i}|}$$
$$= \frac{\sqrt{3^2 + (-7)^2}}{\sqrt{5^2 + (-2)^2}}$$
$$= \frac{\sqrt{58}}{\sqrt{29}}$$
$$= \sqrt{2}$$
$$\arg z = \tan^{-1}\left(-\frac{7}{3}\right) - \tan^{-1}\left(-\frac{2}{5}\right)$$
$$= -\frac{1}{4}\pi$$
$$z = \sqrt{2}\text{e}^{-\frac{1}{4}\text{i}\pi}$$

c $$\frac{z}{w} = \frac{\sqrt{2}\text{e}^{-\frac{1}{4}\text{i}\pi}}{5\text{e}^{\text{i}\frac{\pi}{6}}}$$
$$= \frac{\sqrt{2}}{5}\text{e}^{\text{i}\pi\left(-\frac{1}{4}-\frac{1}{6}\right)}$$
$$= \frac{\sqrt{2}}{5}\text{e}^{-\frac{5}{12}\text{i}\pi}$$

8 $z = a + b\text{i}$

$a^2 + b^2 = 5^2 = 25$ ....[1]

$$\tan\left(\frac{\pi}{6}\right) = \frac{b}{a}$$
$$\frac{\sqrt{3}}{3} = \frac{b}{a}$$
$$a\sqrt{3} = 3b$$

$a = \sqrt{3}b$ ....[2]

Substituting [2] into [1]:

$$\left(\sqrt{3}b\right)^2 + b^2 = 25$$
$$4b^2 = 25$$
$$b^2 = \frac{25}{4}$$
$$b = \pm\sqrt{\frac{25}{4}} = \pm\frac{5}{2}$$

but $\arg z > 0$, so $b > 0$

$$b = \frac{5}{2}$$
$$a = \sqrt{3}b = \frac{5\sqrt{3}}{2}$$

9 **a** $$z = r(\cos\theta + \text{i}\sin\theta)$$
$$z^* = r(\cos\theta - \text{i}\sin\theta)$$
$$zz^* = r^2(\cos\theta + \text{i}\sin\theta)(\cos\theta - \text{i}\sin\theta)$$
$$= r^2(\cos^2\theta - \text{i}^2\sin^2\theta)$$
$$= r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2$$

b
$$\begin{aligned} z &= r(\cos\theta + \mathrm{i}\sin\theta) \\ z^* &= r(\cos\theta - \mathrm{i}\sin\theta) \\ \frac{z}{z^*} &= \frac{r(\cos\theta + \mathrm{i}\sin\theta)}{r(\cos\theta - \mathrm{i}\sin\theta)} \\ &= \frac{r(\cos\theta + \mathrm{i}\sin\theta)(\cos\theta + \mathrm{i}\sin\theta)}{r(\cos\theta - \mathrm{i}\sin\theta)(\cos\theta + \mathrm{i}\sin\theta)} \\ &= \frac{\cos^2\theta + 2\mathrm{i}\sin\theta\cos\theta + \mathrm{i}^2\sin^2\theta}{\cos^2\theta - \mathrm{i}^2\sin^2\theta} \\ &= \frac{\cos^2\theta - \sin^2\theta + (2\sin\theta\cos\theta)\mathrm{i}}{\cos^2\theta + \sin^2\theta} \\ &= \cos 2\theta + \mathrm{i}\sin 2\theta \end{aligned}$$

10 **a**
$$\begin{aligned} z &= r(\cos\theta + \mathrm{i}\sin\theta) \\ z^2 &= r^2(\cos\theta + \mathrm{i}\sin\theta)(\cos\theta + \mathrm{i}\sin\theta) \\ &= r^2\left(\cos^2\theta + 2\mathrm{i}\sin\theta\cos\theta + \mathrm{i}^2\sin^2\theta\right) \\ &= r^2\left(\cos^2\theta - \sin^2\theta + \mathrm{i}\sin 2\theta\right) \\ &= r^2(\cos 2\theta + \mathrm{i}\sin 2\theta) \end{aligned}$$

b
$$\begin{aligned} z &= r(\cos\theta + \mathrm{i}\sin\theta) \\ z - \frac{1}{z} &= r(\cos\theta + \mathrm{i}\sin\theta) - \frac{1}{r(\cos\theta + \mathrm{i}\sin\theta)} \\ &= r(\cos\theta + \mathrm{i}\sin\theta) - \frac{\cos\theta - \mathrm{i}\sin\theta}{r(\cos\theta + \mathrm{i}\sin\theta)(\cos\theta - \mathrm{i}\sin\theta)} \\ &= r(\cos\theta + \mathrm{i}\sin\theta) - \frac{\cos\theta - \mathrm{i}\sin\theta}{r(\cos^2\theta + \sin^2\theta)} \\ &= r(\cos\theta + \mathrm{i}\sin\theta) - \frac{1}{r}\cos\theta + \frac{1}{r}\sin\theta\mathrm{i} \end{aligned}$$

EXERCISE 11D

1 **a** $(-\mathrm{i})^3 + (-\mathrm{i})^2 + (-\mathrm{i}) + k = 0$

$$\mathrm{i}^2(-\mathrm{i}) + \mathrm{i}^2 - \mathrm{i} + k = 0$$
$$\mathrm{i} - 1 + \mathrm{i} + k = 0$$
$$k = 1$$

b

> Always remember that, if z is a root, then z^* will also be a root, provided the polynomial has a degree of at least two.

$z_2 = (-\mathrm{i})^* = \mathrm{i}$

$z - \mathrm{i}$ and $z + \mathrm{i}$ are factors of $\mathrm{f}(z) = z^3 + z^2 + z + 1$.

Dividing:

$$\begin{array}{r} z + 1 \\ z^2 + 0z + 1 \overline{\big) z^3 + z^2 + z + 1} \\ \underline{z^3 + 0z^2 + z} \\ z^2 + 1 \\ \underline{z^2 + 1} \\ 0 \end{array}$$

$z^3 + z^2 + z + 1 = (z^2 + 1)(z + 1)$

$z = -1$ is the third root and is real.

> Remember that you can use long division to find the third factor. The examples in the coursebook use an expansion technique, but mention that long division is possible. The alternative method has been used here, so that you can see it in action.

2 **a** $(z - 5)^3 = 8$

$$z - 5 = \sqrt[3]{8}$$
$$= 2\sqrt[3]{1}$$

$$z - 5 = 2 \times 1, \qquad 2 \times \frac{-1 + \mathrm{i}\sqrt{3}}{2}, \qquad 2 \times \frac{-1 - \mathrm{i}\sqrt{3}}{2}$$
$$z = 7, \qquad 5 + (-1 + \mathrm{i}\sqrt{3}), \qquad 5 + (-1 - \mathrm{i}\sqrt{3})$$
$$z = 7, \qquad 4 + \mathrm{i}\sqrt{3}, \qquad 4 - \mathrm{i}\sqrt{3}$$

c $(2z + 3)^3 = \frac{1}{64}$

$$2z + 3 = \sqrt[3]{\frac{1}{64}}$$
$$= \frac{1}{4}\sqrt[3]{1}$$

$$2z + 3 = \frac{1}{4} \times 1, \qquad \frac{1}{4} \times \frac{-1 + \mathrm{i}\sqrt{3}}{2}, \qquad \frac{1}{4} \times \frac{-1 - \mathrm{i}\sqrt{3}}{2}$$
$$2z = -\frac{11}{4}, \qquad \frac{-1 + \mathrm{i}\sqrt{3}}{8} - 3, \qquad \frac{-1 - \mathrm{i}\sqrt{3}}{8} - 3$$
$$z = -\frac{11}{8}, \qquad -\frac{25}{16} + \frac{\mathrm{i}\sqrt{3}}{16}, \qquad -\frac{25}{16} - \frac{\mathrm{i}\sqrt{3}}{16}$$

3 $2z^2 + z + 3 = 0$

$$z = \frac{-1 \pm \sqrt{1^2 - 4(2)(3)}}{2(2)}$$
$$= \frac{-1 \pm \sqrt{-23}}{4}$$
$$= \frac{-1 \pm \mathrm{i}\sqrt{23}}{4}$$

$z_1 = \frac{-1 + i\sqrt{23}}{4}, \quad z_2 = \frac{-1 - i\sqrt{23}}{4}$

$\arg z_1 = \pi - \tan^{-1}\left(\frac{\sqrt{23}}{1}\right) = 1.78$

$\arg z_2 = -\arg z_1$ because the roots are complex conjugates.

$$|z_1| = |z_2| = \sqrt{\left(-\frac{1}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2}$$
$$= \sqrt{\frac{1}{16} + \frac{23}{16}}$$
$$= \sqrt{\frac{24}{16}} = \frac{2\sqrt{6}}{4}$$
$$= \frac{\sqrt{6}}{2}$$

$z_1 = \frac{\sqrt{6}}{2}(\cos 1.78 + i \sin 1.78)$

$z_2 = \frac{\sqrt{6}}{2}(\cos(-1.78) + i \sin(-1.78))$

Remember that if z_1 and z_2 are complex conjugate roots of the same equation then $\arg(z_1) = -\arg(z_2)$.

4 Dividing:

$$\begin{array}{r l}
 & z^2 + 25 \\
z - 3 \big) & z^3 - 3z^2 + 25z - 75 \\
 & \underline{z^3 - 3z^2} \\
 & 25z - 75 \\
 & \underline{25z - 75} \\
 & 0
\end{array}$$

$z^3 - 3z^2 + 25z - 75 = (z - 3)(z^2 + 25)$

$z = 3$

or $z = \pm\sqrt{-25} = \pm 5i$

5 $x^2 + 2xyi + i^2y^2 = 55 + 48i$

Equating real parts:

$x^2 - y^2 = 55$ ....[1]

Equating imaginary parts:

$2xy = 48$

$y = \frac{24}{x}$ ....[2]

Substituting into [1]:

$$x^2 - \left(\frac{24}{x}\right)^2 = 55$$
$$(x^2)^2 - 576 = 55x^2$$
$$(x^2)^2 - 55x^2 - 576 = 0$$
$$(x^2 - 64)(x^2 + 9) = 0$$

$x = \pm 8$ or $x = \pm 3i$

x is real and positive, so $x = 8$

$y = \frac{24}{8} = 3$

6 Dividing:

$$\begin{array}{r l}
 & z^2 - 6z + 10 \\
2z + 1 \big) & 2z^3 - 11z^2 + 14z + 10
\end{array}$$

$$2z^3 + z^2$$
$$-12z^2 + 14z$$
$$-12z^2 - 6z$$
$$20z + 10$$
$$20z + 10$$
$$0$$

$$2z^3 - 11z^2 + 14z + 10 = (2z+1)(z^2 - 6z + 10)$$
$$z = -\frac{1}{2}$$

or

$$(z-3)^2 - 9 + 10 = 0$$
$$(z-3)^2 = -1$$
$$z - 3 = \pm \mathrm{i}$$
$$z = 3 \pm \mathrm{i}$$
$$z = -\frac{1}{2}, 3+\mathrm{i}, 3-\mathrm{i}$$

7 $z = 3\mathrm{i}$ is a root, so $z = -3\mathrm{i}$ is also a root.

$(z - 3\mathrm{i})(z + 3\mathrm{i}) = z^2 + 9$ is a factor of $z^4 - 2z^3 + 14z^2 - 18z + 45 = 0$

Dividing:

$$
\begin{array}{r}
z^2 - 2z + 5 \\
z^2 + 0z + 9 \,\overline{\big)\, z^4 - 2z^3 + 14z^2 - 18z + 45} \\
\underline{z^4 + 0z^3 + 9z^2} \qquad\qquad \\
-2z^3 + 5z^2 - 18z \qquad \\
\underline{-2z^3 + 0z^2 - 18z} \qquad \\
5z^2 + 0z + 45 \\
\underline{5z^2 + 0z + 45} \\
0
\end{array}
$$

$$z^4 - 2z^3 + 14z^2 - 18z + 45 = (z^2 + 9)(z^2 - 2z + 5)$$

$z = \pm 3\mathrm{i}$ as before

or $z^2 - 2z + 5 = 0$

$$(z-1)^2 - 1 + 5 = 0$$
$$(z-1)^2 = -4$$
$$(z-1) = \pm\sqrt{-4}$$
$$z - 1 = \pm 2\mathrm{i}$$
$$z = 1 \pm 2\mathrm{i}$$

$$z = 3\mathrm{i}, -3\mathrm{i}, 1+2\mathrm{i}, 1-2\mathrm{i}$$

8 **b**

$$(x + iy)^2 = 7 + (6\sqrt{2})\mathrm{i}$$
$$x^2 + 2xy\mathrm{i} + \mathrm{i}^2 y^2 = 7 + (6\sqrt{2})\mathrm{i}$$

Equating real and imaginary parts:

$x^2 - y^2 = 7$ ....[1]

$2xy = 6\sqrt{2}$

$y = \dfrac{3\sqrt{2}}{x}$ ....[2]

Substituting into [1]:

$$x^2 - \left(\frac{3\sqrt{2}}{x}\right)^2 = 7$$
$$(x^2)^2 - 18 = 7x^2$$
$$(x^2)^2 - 7x^2 - 18 = 0$$
$$(x^2 - 9)(x^2 + 2) = 0$$

$x = \pm 3$ or

x and y are real numbers.

$x = 3, y = \sqrt{2}$

or

$x = -3, y = -\sqrt{2}$

Square roots are:

$\pm(3 + i\sqrt{2})$

f Square roots are of the form

$$\begin{aligned} z &= re^{i\theta} \\ z^2 &= r^2 e^{2i\theta} \\ r^2 e^{2i\theta} &= \frac{1}{2} e^{\frac{i\pi}{2}} \\ r^2 &= \frac{1}{2} \\ r &= \frac{\sqrt{2}}{2} \\ 2\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{4} \end{aligned}$$

Square roots are

$z = \pm \frac{\sqrt{2}}{2} e^{\frac{i\pi}{4}}$

But

$$\begin{aligned} -e^{i\frac{\pi}{4}} &= (-1)e^{\frac{i\pi}{4}} \\ &= e^{-i\pi} e^{\frac{i\pi}{4}} = e^{\frac{-3i\pi}{4}} \end{aligned}$$

So $z = \frac{\sqrt{2}}{2} e^{\frac{i\pi}{4}}$ or $\frac{\sqrt{2}}{2} e^{\frac{-3i\pi}{4}}$

> Remember that your argument needs to lie between $-\pi$ and π and the modulus needs to be positive. You can use the fact that $e^{\pm i\pi} = -1$ to change the sign of a complex number.

9 **a** Complex conjugate = 2−i.

b

$$\begin{aligned} (2 + i)^3 - 12(2 + i)^2 + p(2 + i) + q = 0 \\ 8 + 12i + 6i^2 + i^3 - 12(4 + 4i + i^2) + 2p + pi + q = 0 \end{aligned}$$

Equating real and imaginary parts:

$8 - 6 - 48 + 12 + 2p + q = 0$

$2p + q = 34$ ....[1]

$12 - 1 - 48 + p = 0$ ....[2]

$p = 37$

Then from [1]:

$$\begin{aligned} 74 + q &= 34 \\ q &= -40 \end{aligned}$$

c

> Keep reminding yourself that if two polynomials are factors of another polynomial then so is their product.

$z^3 - 12z^2 + 37z - 40 = 0$

$(z - (2 + i))(z - (2 - i))$

$= z^2 - z(2 - i + 2 + i) + (4 + 1)$

$= z^2 - 4z + 5$

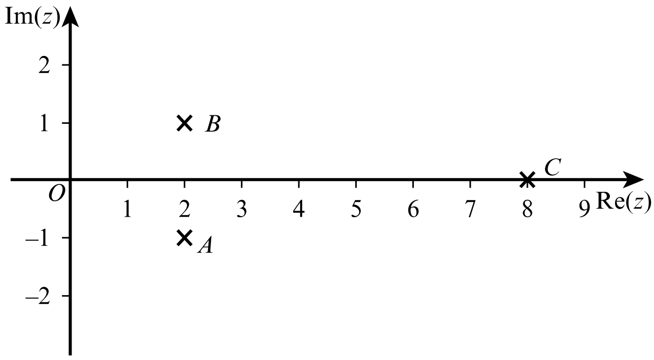
is a factor of $z^3 - 12z^2 + 37z - 40$

Dividing:

$$\begin{array}{r} z-8 \\ z^2-4z+5 \overline{\big) z^3-12z^2+37z-40} \\ \underline{z^3-4z^2+5z} \\ -8z^2+32z-40 \\ \underline{-8z^2+32z-40} \\ 0 \end{array}$$

$$z^3-12z^2+37z-40=(z-8)(z^2-4z+5)$$

$$z=8, 2+\mathrm{i},\ 2-\mathrm{i}$$



10 $z^4-z^3+20z^2-16z+64=(z^2+a)(z^2+bz+4)$

Considering the constant term:

$$4a=64$$
$$a=16$$

Considering the number coefficients of z^3:

$b=-1$

$$z^4-z^3+20z^2-16z+64=(z^2+16)(z^2-z+4)$$
$$z=\pm 4\mathrm{i}$$

or

$$z^2-z+4=0$$
$$\left(z-\frac{1}{2}\right)^2-\frac{1}{4}+4=0$$
$$\left(z-\frac{1}{2}\right)^2=-\frac{15}{4}$$
$$z=\frac{1}{2}\pm\frac{\mathrm{i}\sqrt{15}}{2}$$

$$z=4\mathrm{i}, -4\mathrm{i},\ \frac{1}{2}+\frac{\mathrm{i}\sqrt{15}}{2},\ \frac{1}{2}-\frac{\mathrm{i}\sqrt{15}}{2}$$

EXERCISE 11E

1 **a** $\arg(z - 2 + 3\text{i})$

$= \arg\{z - (2 - 3\text{i})\} = \dfrac{\pi}{12}$

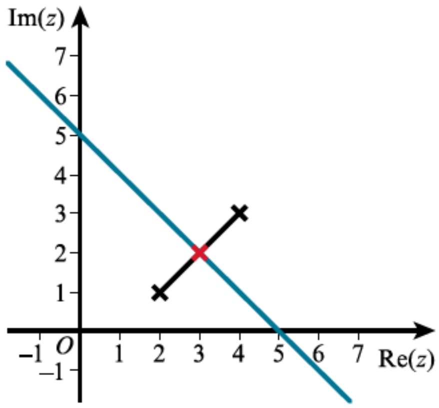
So this is a half line from $(2, -3)$ at an angle of $\dfrac{\pi}{12}$ radians.

c $|z + 6 - \text{i}| = 7$

$|z - (-6 + \text{i})| = 7$

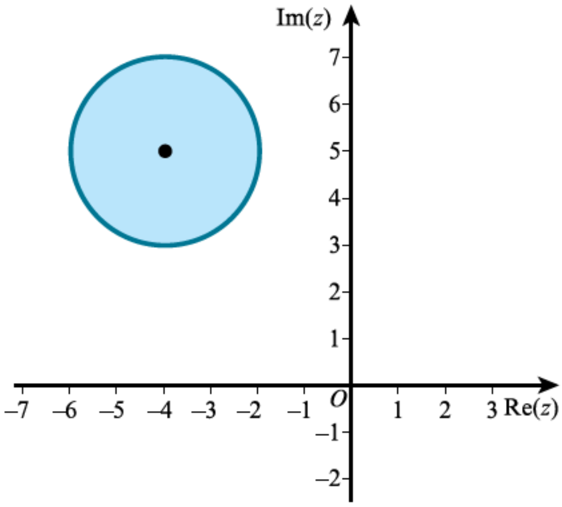
So this is a circle, radius 7, centre $(-6, 1)$.

2 **a** Perpendicular bisector of $(2, 1)$ and $(4, 3)$.



c $|z - (-4 + 5\text{i})| \leqslant 2$

This is the inside of a circle, radius 2, centre $(-4, 5)$.

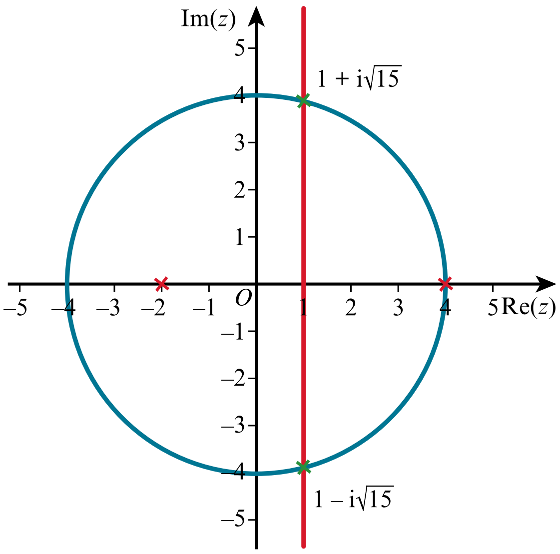


Note that part **c** uses an inequality. This means that the locus required is, in fact, all circles with radius less than or equal to 2, so you need to shade the entire disc.

3 The loci required are:

For $|z| = 4$, a circle, centre $(0, 0)$ and radius 4.

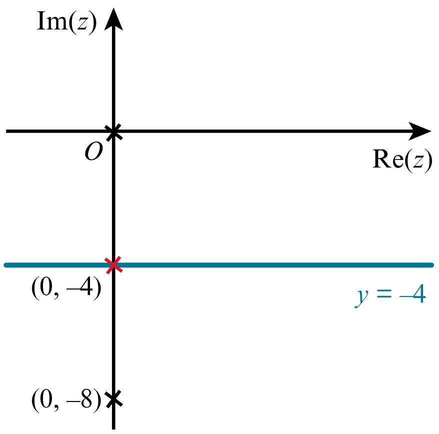
For $|z + 2| = |z - 4|$, the perpendicular bisector of the points $x = -2$ and $x = 4$. This is the line with equation $x = 1$.



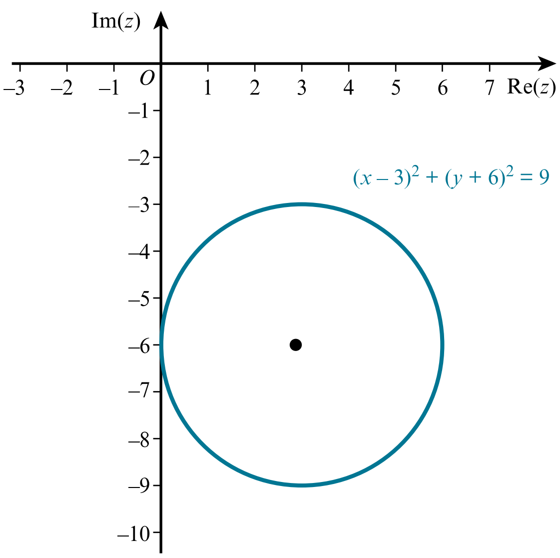
The loci intersect when:

$$x^2 + y^2 = 16$$
$$x = 1$$
$$1 + y^2 = 16$$
$$y^2 = 15$$
$$y = \pm\sqrt{15}$$
$$z = 1 \pm \text{i}\sqrt{15}$$

4 This is the perpendicular bisector of the points $(0, 0)$ and $(0, -8)$. It is the line $y = -4$.



5 This is the circle with radius 3 and centre $(3, -6)$.



The Cartesian equation is:

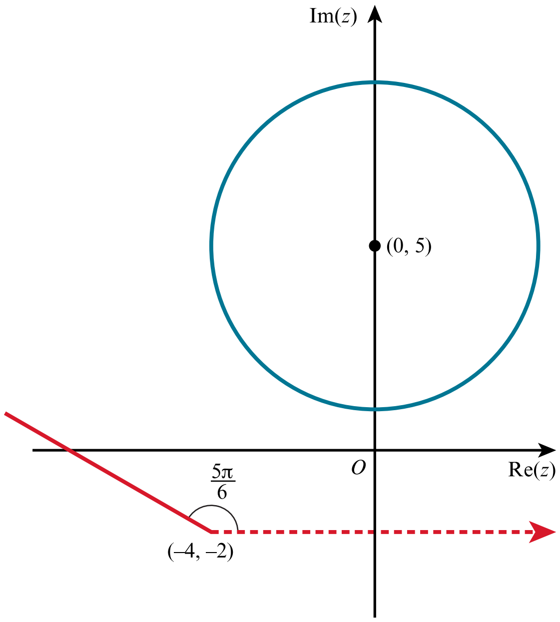
$(x-3)^2+(y+6)^2=9$

6 $\arg(z-(-4-2\text{i}))=\frac{5\pi}{6}$

This is a half line from $(-4,-2)$ at an angle $\frac{5\pi}{6}$.

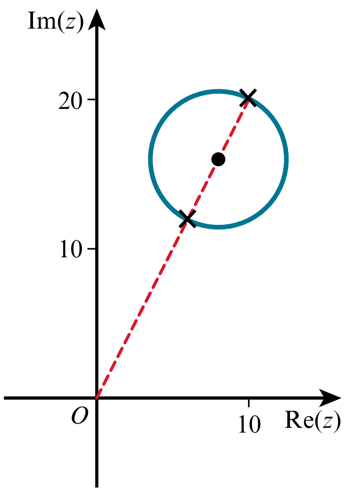
$|z-5\text{i}|=4$

This is a circle, radius 4, centre $(0, 5)$.



The diagram shows that the two loci do not intersect. So there is no complex number, z, that satisfies both loci.

7 This is a circle, radius $2\sqrt{5}$, centre $(8, 16)$.



The centre of the circle lies at a distance

$\sqrt{8^2+16^2}$
$=\sqrt{64+256}$
$=\sqrt{320}$
$=\sqrt{64}\sqrt{5}$
$=8\sqrt{5}$
from $(0, 0)$.

The greatest and least values of $|z|$ are, therefore,

$8\sqrt{5}\pm$ (radius)

$=8\sqrt{5}\pm 2\sqrt{5}$

$=6\sqrt{5}$ (minimum)

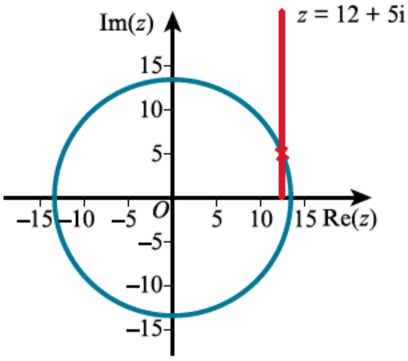
or

$10\sqrt{5}$ (maximum)

The diagram is vital here. You can see, very easily, where the greatest and least values lie.

8 $|z| = 13$: Circle centre $(0,\ 0)$ and radius 13.

$\arg(z-12) = \frac{\pi}{2}$: Half line from $(12,\ 0)$ and angle $\frac{\pi}{2}$, which is the half line with equation $x = 12$ that lies above the real axis only.

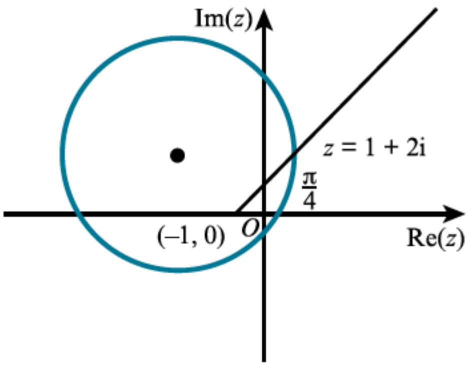


The loci intersect when:

$$x^2 + y^2 = 169$$
$$x = 12$$
$$144 + y^2 = 169$$
$$y^2 = 25$$
$$y = \pm 5$$
$$y > 0$$
$$y = 5$$
$$z = 12 + 5\text{i}$$

9 $|z + 3 - 2\text{i}| = 4$: Circle, centre $(-3,\ 2)$ and radius 4.

$\arg(z+1) = \frac{\pi}{4}$: Half line from -1 with angle $\frac{\pi}{4}$, which is the line with gradient 1, from the point $x = -1$ above the real axis, i.e. $y = x + 1$.



The loci intersect when:

$$(x+3)^2 + (y-2)^2 = 16$$

Substituting $y = x + 1$:

$$(x+3)^2 + (x-1)^2 = 16$$
$$2x^2 + 4x + 10 = 16$$
$$2x^2 + 4x - 6 = 0$$
$$x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$

$x = 1$ or $x = -3$

$y = 2$ or $y = -2$

$y > 0$

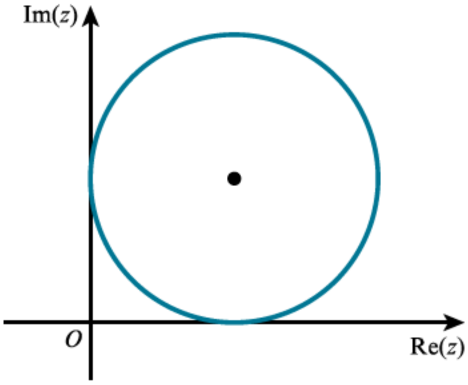
$x = 1$ and $y = 2$

$z = 1 + 2\text{i}$

10 **a** Circle, centre $(5, 5)$, radius 5.

$(x - 5)^2 + (y - 5)^2 = 25$

b



Note that, because the radius is 5 and both the x and y-coordinates of the centre are also 5, the circle has both axes as tangents.

c Given that the circle is neatly contained between the real and imaginary axes, the diagram shows that the least and greatest values of $\arg z$ are 0 and $\frac{\pi}{2}$ respectively.

END-OF-CHAPTER REVIEW EXERCISE 11

P3 **This exercise is for Pure Mathematics 3 students only.**

1 **a** $\frac{5-2i}{1+3i}$

$= \frac{(5-2i)(1-3i)}{(1+3i)(1-3i)}$

$= \frac{5-15i-2i+6i^2}{1-9i^2}$

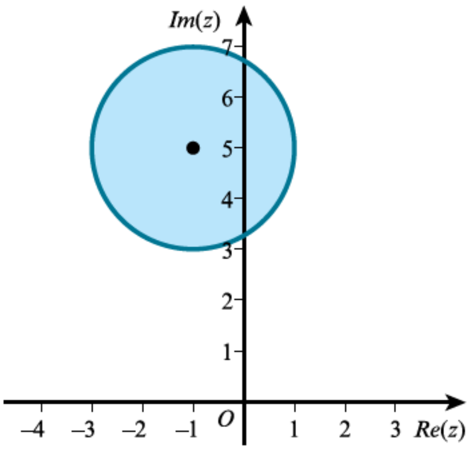
$= \frac{-1-17i}{1+9}$

$= -\frac{1}{10} - \frac{17i}{10}$

Remember that this method always involves the expansion of a difference between two squares in the denominator. You can use the fact that $x^2 - a^2 = (x-a)(x+a)$ to save some working.

b
$$\begin{aligned} w^2 - 2w + 26 &= 0 \\ (w-1)^2 - 1 + 26 &= 0 \\ (w-1)^2 &= -25 \\ w - 1 &= \pm 5i \\ w &= 1 \pm 5i \end{aligned}$$

c $|z-(-1+5i)| = 2$ would give a circle, centre $(-1, 5)$ and radius 2. With the inequality you need the inside of the circle.



2 **a**
$$\begin{aligned} z &= k - 6i \\ z^* &= k + 6i \\ zz^* &= (k-6i)(k+6i) \\ &= k^2 - 36i^2 \\ &= k^2 + 36 \end{aligned}$$

$$\begin{aligned} \frac{z}{z^*} &= \frac{k-6i}{k+6i} \\ &= \frac{(k-6i)(k-6i)}{(k+6i)(k-6i)} \\ &= \frac{k^2 - 12ki + 36i^2}{k^2 - 36i^2} \\ &= \frac{k^2 - 36 - 12ki}{k^2 + 36} \\ &= \frac{k^2 - 36}{k^2 + 36} - \frac{12ki}{k^2 + 36} \end{aligned}$$

b $u = 4e^{\frac{5i\pi}{12}}$

$w = 2e^{i\pi}$

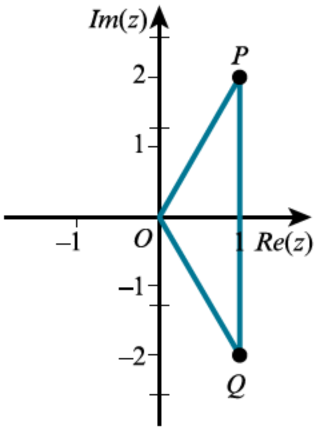
$$uw = 8e^{i\pi\left(\frac{5}{12}+1\right)}$$
$$= 8e^{\frac{17}{12}i\pi}$$
$$= 8e^{\frac{17}{12}i\pi - 2\pi}$$
$$= 8e^{-\frac{7}{12}i\pi}$$

Notice how subtracting 2π from the argument – pushes the argument back into the correct interval. This doesn't change the actual complex number, because it is exactly one sweep around the Argand diagram.

$$\frac{u}{w} = 2e^{i\pi\left(\frac{5}{12}-1\right)}$$
$$= 2e^{-\frac{7}{12}i\pi}$$

3 **a** $w = 1 + 2i$

$w^* = 1 - 2i$



OPQ is an isosceles triangle, because $|w| = |w^*|$.

b
$$\frac{u}{w} = \frac{-3-i}{1+2i}$$
$$= \frac{-(3+i)(1-2i)}{(1+2i)(1-2i)}$$
$$= \frac{-3+6i-i+2i^2}{1-4i^2}$$
$$= \frac{-5+5i}{1+4}$$
$$= -1 + i$$
$$\left|\frac{u}{w}\right| = \sqrt{1^2+1^2} = \sqrt{2}$$
$$\arg(-1+i) = \pi - \tan^{-1}(1)$$
$$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$
$$\frac{u}{w} = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

4 **a** $z^* = 2 + 5i$

$2 + 5i = 2x + 1 + (4x + y)i$

Equating real parts:

$2x + 1 = 2$

$$x = \frac{1}{2}$$

Equating imaginary parts:

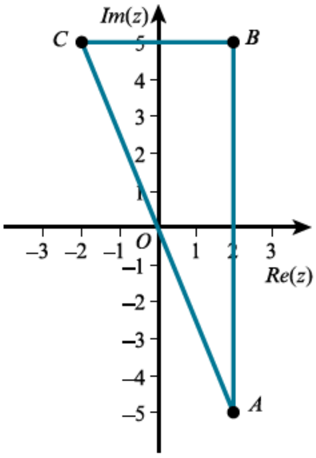
$$4x + y = 5$$
$$2 + y = 5$$
$$y = 3$$

b $z = 2 - 5\text{i}$

$z^* = 2 + 5\text{i}$

$-z = -2 + 5\text{i}$

Notice that $-z$ is the reflection of z^* in the imaginary axis. Also z and z^* have the same real part, so z^* lies vertically above z on the Argand diagram. Similarly z^* and $-z$ have the same imaginary part, so $-z$ lies directly to the left of z^* on the Argand diagram. Given that the sides joining each of these pairs of points lie, respectively, vertically and horizontally, the triangle must be right-angled.



c i $\dfrac{z^*}{-z} = \dfrac{2+5\text{i}}{-2+5\text{i}}$

$$= \frac{(2+5\text{i})(-2-5\text{i})}{(-2+5\text{i})(-2-5\text{i})}$$
$$= \frac{-4-20\text{i}-25\text{i}^2}{4-25\text{i}^2}$$
$$= \frac{21-20\text{i}}{29}$$
$$= \frac{21}{29} - \frac{20\text{i}}{29}$$

ii $\left|\dfrac{21}{29} - \dfrac{20\text{i}}{29}\right| = \sqrt{\left(\dfrac{21}{29}\right)^2 + \left(\dfrac{20}{29}\right)^2}$

$$= \frac{1}{29}\sqrt{21^2 + 20^2}$$
$$= \frac{\sqrt{841}}{29} = 1$$

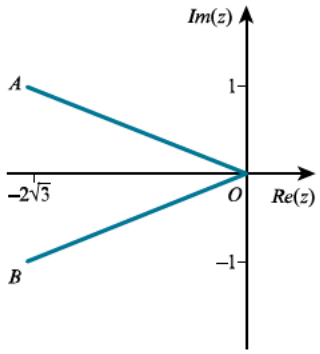
$$\arg\left(\frac{21}{29} - \frac{20\text{i}}{29}\right) = \tan^{-1}\left(\frac{20}{-21}\right) = -0.761$$

$$\frac{21}{29} - \frac{20\text{i}}{29} = \cos(-0.761) + \text{i}\sin(-0.761)$$

5 a $z^2 + (4\sqrt{3})z + 13 = 0$

$$(z + 2\sqrt{3})^2 - 12 + 13 = 0$$
$$(z + 2\sqrt{3})^2 = -1$$
$$z + 2\sqrt{3} = \pm\text{i}$$
$$z = -2\sqrt{3} \pm \text{i}$$

b



The vectors are reflections in the real axis.

c

Two complex conjugates will always have the same modulus.

$$z_1 = -2\sqrt{3} + \mathrm{i}$$
$$z_2 = -2\sqrt{3} - \mathrm{i}$$
$$|z_1| = |z_2| = \sqrt{\left(2\sqrt{3}\right)^2 + 1^2} = \sqrt{13}$$
$$\arg z_1 = \pi - \tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) = 2.86$$
$$\arg z_2 = -\arg z_1 = -2.86$$

6 a

$$z = 4\sqrt{3} - 4\mathrm{i}$$
$$|z| = \sqrt{\left(4\sqrt{3}\right)^2 + (-4)^2} = \sqrt{48 + 16} = \sqrt{64} = 8$$
$$\arg z = \tan^{-1}\left(\frac{-4}{4\sqrt{3}}\right) = -\frac{\pi}{6}$$

b

$$z = 8\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right) = 8\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}$$
$$w = 2\sqrt{2}\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right) = 2\sqrt{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{12}}$$
$$\frac{z}{w} = \frac{8\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}}{2\sqrt{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{12}}} = 2\sqrt{2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}-\frac{\mathrm{i}\pi}{12}}$$
$$= 2\sqrt{2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}$$

Remember that $r(\cos\theta + \mathrm{i}\sin\theta) \equiv r\mathrm{e}^{\mathrm{i}\theta}$.

7 a

$$w^* - 2 - 2\mathrm{i} = 3\mathrm{i}w$$
$$w = x + \mathrm{i}y$$
$$(x - \mathrm{i}y) - 2 - 2\mathrm{i} = 3\mathrm{i}(x + \mathrm{i}y)$$
$$x - \mathrm{i}y - 2 - 2\mathrm{i} = 3x\mathrm{i} - 3y$$

Equating real parts:

$$x - 2 = -3y$$
$$x + 3y = 2 \ldots\ldots\ldots\ldots [1]$$

Equating imaginary parts:

$$-y - 2 = 3x$$
$$3x + y = -2 \ldots\ldots\ldots [2]$$

$3 \times [1]$:

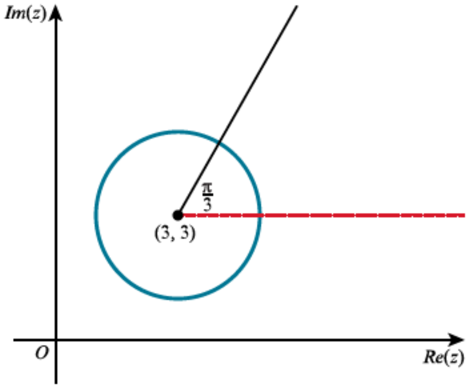
$$8y = 8$$
$$y = 1$$

Substituting into [1]:

$$x + 3 = 2$$
$$x = -1$$
$$w = -1 + \mathrm{i}$$

b **i** $|z-3-3\mathrm{i}|=2$: Circle, centre $(3, 3)$ and radius 2.

$\arg(z-3-3\mathrm{i})=\frac{\pi}{3}$: Half line from $(3, 3)$ with angle $\frac{\pi}{3}$.



ii Half line gradient $=\tan\frac{\pi}{3}=\sqrt{3}$

Equation of the half line:

$$y-3=\sqrt{3}(x-3)$$
$$y=\sqrt{3}x+3-3\sqrt{3} \text{ .... [1]}$$

Equation of the circle:

$$(x-3)^2+(y-3)^2=4 \text{ .... [2]}$$

Substituting into [1]:

$$\begin{aligned}(x-3)^2+\left(\sqrt{3}x-3\sqrt{3}\right)^2&=4\\ x^2-6x+9+3x^2-18x+27&=4\\ 4x^2-24x+32&=0\\ x^2-6x+8&=0\\ (x-4)(x-2)&=0\end{aligned}$$

$x=2$ or $x=4$

Half line, so $x>3$

$$x=4$$
$$y=\sqrt{3}(4)+3-3\sqrt{3}=3+\sqrt{3}$$
$$z=4+(3+\sqrt{3})\mathrm{i}$$

8 **a**

Remember that by solving Question **8a** you are actually finding the square root of $7-6\sqrt{2}\mathrm{i}$.

$$\begin{aligned}(x+\mathrm{i}y)^2&=7-\left(6\sqrt{2}\right)\mathrm{i}\\ x^2+2xy\mathrm{i}+\mathrm{i}^2y^2&=7-\left(6\sqrt{2}\right)\mathrm{i}\\ x^2-y^2+2xy\mathrm{i}&=7-\left(6\sqrt{2}\right)\mathrm{i}\end{aligned}$$

Equating real parts:

$$x^2-y^2=7 \text{ ....[1]}$$

Equating imaginary parts:

$$\begin{aligned}2xy&=-6\sqrt{2}\\ xy&=-3\sqrt{2}\\ y&=-\frac{3\sqrt{2}}{x} \quad [2]\end{aligned}$$

Substituting into [1]:

$$\begin{aligned}x^2-\left(-\frac{3\sqrt{2}}{x}\right)^2&=7\\ x^2-\frac{18}{x^2}&=7\\ \left(x^2\right)^2-7x^2-18&=0\\ \left(x^2-9\right)\left(x^2+2\right)&=0\end{aligned}$$

$x^2=9$ or $x^2=-2$

$x = \pm 3$ (no real roots to $x^2 = -2$)

$x = 3, y = -\sqrt{2}$

$x = -3, y = \sqrt{2}$

b **i** $f(z) = 2z^3 - 4z^2 - 5z - 3$

$f(3) = 54 - 36 - 15 - 3 = 0$

$z - 3$ is a factor of $f(z)$ by the factor theorem.

ii Dividing:

$$\begin{array}{r} 2z^2 + 2z + 1 \\ z - 3 \overline{\big) 2z^3 - 4z^2 - 5z - 3} \\ \underline{2z^3 - 6z^2} \qquad\qquad \\ 2z^2 - 5z \qquad \\ \underline{2z^2 - 6z} \qquad \\ z - 3 \\ \underline{z - 3} \\ 0 \end{array}$$

$2z^3 - 4z^2 - 5z - 3 = (z - 3)(2z^2 + 2z + 1)$

$2z^2 + 2z + 1 = 0 \quad \text{or} \quad z = 3$

$$z = \frac{-2 \pm \sqrt{2^2 - 4(2)(1)}}{2(2)}$$

$$z = \frac{-2 \pm \sqrt{-4}}{4}$$

$$= \frac{-2 \pm 2\text{i}}{4}$$

$$z = 3, -\frac{1}{2} + \frac{1}{2}\text{i}, -\frac{1}{2} - \frac{1}{2}\text{i}$$

9 **a** $z_2 = \dfrac{21 + \text{i}}{z_1}$

$$= \frac{21 + \text{i}}{5 - 3\text{i}}$$

$$= \frac{(21 + \text{i})(5 + 3\text{i})}{(5 - 3\text{i})(5 + 3\text{i})}$$

$$= \frac{105 + 68\text{i} + 3\text{i}^2}{25 + 9}$$

$$= \frac{102 + 68\text{i}}{34}$$

$$= 3 + 2\text{i}$$

b $(3z + 1)^3 = -27$

$(3z + 1)^3 = -3^3(1)$

$3z + 1 = -3\sqrt[3]{1}$

$$3z + 1 = -3 \times 1, \ -3 \times \left(-\frac{1}{2} + \frac{\text{i}\sqrt{3}}{2}\right), \ -3 \times \left(-\frac{1}{2} - \frac{\text{i}\sqrt{3}}{2}\right)$$

$$3z + 1 = -3, \ \frac{3}{2} - \frac{3\sqrt{3}\text{i}}{2}, \ \frac{3}{2} + \frac{3\sqrt{3}\text{i}}{2}$$

$$3z = -4, \ \frac{1}{2} - \frac{3\sqrt{3}\text{i}}{2}, \ \frac{1}{2} + \frac{3\sqrt{3}\text{i}}{2}$$

$$z = -\frac{4}{3}, \ \frac{1}{6} - \frac{\sqrt{3}\text{i}}{2}, \ \frac{1}{6} + \frac{\sqrt{3}\text{i}}{2}$$

Remember that the complex cube roots of 1 are 1, $-\dfrac{1}{2} + \dfrac{\text{i}\sqrt{3}}{2}$ and $-\dfrac{1}{2} - \dfrac{\text{i}\sqrt{3}}{2}$.

10 **a** **i** $f(w) = 2w^4 + 5w^3 - 2w^2 + w - 6$

$f(-3) = 2(-3)^4 + 5(-3)^3 - 2(-3)^2 + (-3) - 6$

$= 162 - 135 - 18 - 3 - 6 = 0$

$w + 3$ is a factor $f(w)$ by the factor theorem.

ii The question gives that $w = 1$ is a root, so $w-1$ is also a factor of $f(w)$ by the factor theorem.

$(w + 3)(w - 1) = w^2 + 2w - 3$ is also a factor of $f(w)$.

Dividing:

$$\begin{array}{r}
2w^2 + w + 2 \\
w^2 + 2w - 3\overline{)\,2w^4 + 5w^3 - 2w^2 + w - 6} \\
\underline{2w^4 + 4w^3 - 6w^2} \qquad\qquad \\
w^3 + 4w^2 + w \quad \\
\underline{w^3 + 2w^2 - 3w} \quad \\
2w^2 + 4w - 6 \\
\underline{2w^2 + 4w - 6} \\
0
\end{array}$$

$$2w^4 + 5w^3 - 2w^2 + w - 6 = 0$$
$$(w^2 + 2w - 3)(2w^2 + w + 2) = 0$$
$$(w + 3)(w - 1)(2w^2 + w + 2) = 0$$

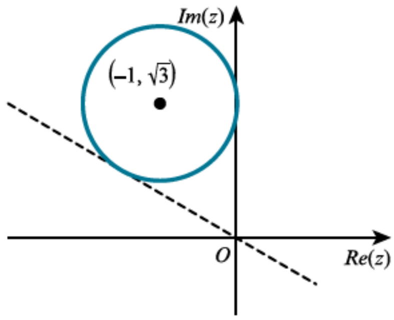
$w = -3$ or $w = 1$

or $w = \dfrac{-1 \pm \sqrt{1^2 - 4(2)(2)}}{2(2)}$

$w = \dfrac{-1 \pm \sqrt{-15}}{4}$

$w = -\dfrac{1}{4} \pm \dfrac{i\sqrt{15}}{4}$

b i Circle, radius 1, centre $(-1, \sqrt{3})$.



The circle has radius 1 and the x-ordinate of the centre is -1. This means that the imaginary axis is a tangent to the circle.

ii Minimum argument occurs when the circle touches the imaginary axis: $\arg z = \dfrac{\pi}{2}$.

iii Maximum argument occurs when the circle touches the dotted line shown in the diagram.

Notice that the argument of the complex number at the centre of the circle is

$\dfrac{\pi}{2} + \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

$= \dfrac{\pi}{2} + \dfrac{\pi}{6}$

Given that the centre lies half way between the imaginary axis and the dotted line, add another $\dfrac{\pi}{6}$.

So the maximum argument is $\dfrac{\pi}{2} + \dfrac{\pi}{6} + \dfrac{\pi}{6} = \dfrac{5\pi}{6}$.

11 a i $z = -\dfrac{3}{2} + \dfrac{\sqrt{7}}{2}i$

$$\left(-\frac{3}{2}+\frac{\sqrt{7}}{2}\mathrm{i}\right)^2+p\left(-\frac{3}{2}+\frac{\sqrt{7}}{2}\mathrm{i}\right)+q=0$$

$$\frac{9}{4}-\frac{3\sqrt{7}}{2}\mathrm{i}+\frac{7}{4}\mathrm{i}^2-\frac{3}{2}p+p\frac{\sqrt{7}}{2}\mathrm{i}+q=0$$

$$\frac{9}{4}-\frac{3\sqrt{7}}{2}\mathrm{i}-\frac{7}{4}-\frac{3}{2}p+p\frac{\sqrt{7}}{2}\mathrm{i}+q=0$$

Equating real parts:

$$\frac{1}{2}-\frac{3}{2}p+q=0$$

$2q-3p=-1$ ....[1]

Equating imaginary parts:

$$-\frac{3\sqrt{7}}{2}+p\frac{\sqrt{7}}{2}=0$$

$$p=3$$

The from [1]:

$$2q-9=-1$$

$$2q=8$$

$$q=4$$

ii $\left|-\frac{3}{2}+\frac{\sqrt{7}}{2}\mathrm{i}\right|$

$$=\sqrt{\left(-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}$$

$$=\sqrt{\frac{9}{4}+\frac{7}{4}}$$

$$=\sqrt{4}$$

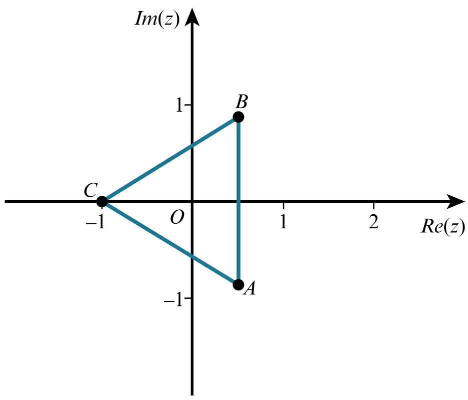
$$=2$$

b i $z^3=-1$

$$z=-1\times 1,\quad -1\times\left(-\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right),\quad -1\times\left(-\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right)$$

$$z=-1,\ \ \frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2},\ \ \frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}$$

ii



The triangle is equilateral.

12 **a** $z = \sqrt{5} - \mathrm{i}$

$z^* = \sqrt{5} + \mathrm{i}$

$$\frac{z}{z^*} = \frac{\sqrt{5} - \mathrm{i}}{\sqrt{5} + \mathrm{i}}$$
$$= \frac{(\sqrt{5} - \mathrm{i})(\sqrt{5} - \mathrm{i})}{(\sqrt{5} + \mathrm{i})(\sqrt{5} - \mathrm{i})}$$
$$= \frac{5 - 2\sqrt{5}\mathrm{i} + \mathrm{i}^2}{5 - \mathrm{i}^2}$$
$$= \frac{4 - 2\sqrt{5}\mathrm{i}}{6}$$
$$= \frac{2}{3} - \frac{\sqrt{5}\mathrm{i}}{3}$$

b

$$\left|\frac{2}{3} - \frac{\sqrt{5}\mathrm{i}}{3}\right| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{\sqrt{5}}{3}\right)^2}$$
$$= \sqrt{\frac{4}{9} + \frac{5}{9}}$$
$$= 1$$

$$\arg\left(\frac{2}{3} - \frac{\sqrt{5}\mathrm{i}}{3}\right) = \tan^{-1}\left(\frac{-\sqrt{5}}{2}\right) = -0.841$$

c

Note that if w and w^* are roots of a polynomial then $(z - w)$ and $z - w^*$ will both be factors. In turn, $(z - w)(z - w^*)$ will also be a factor.

$$\left[z - \left(\frac{2}{3} - \frac{\sqrt{5}\mathrm{i}}{3}\right)\right]\left[z - \left(\frac{2}{3} + \frac{\sqrt{5}\mathrm{i}}{3}\right)\right] = 0$$
$$z^2 - \left(\frac{2}{3} - \frac{\sqrt{5}\mathrm{i}}{3} + \frac{2}{3} + \frac{\sqrt{5}\mathrm{i}}{3}\right)z - \left(\frac{2}{3} - \frac{\sqrt{5}\mathrm{i}}{3}\right)\left(\frac{2}{3} + \frac{\sqrt{5}\mathrm{i}}{3}\right) = 0$$
$$z^2 - \frac{4}{3}z + \left(\frac{4}{9} - \frac{5\mathrm{i}^2}{9}\right) = 0$$
$$z^2 - \frac{4}{3}z + 1 = 0$$
$$3z^2 - 4z + 3 = 0$$

13 **a**

$$\frac{k - 4\mathrm{i}}{2k - \mathrm{i}} = \frac{(k - 4\mathrm{i})(2k + \mathrm{i})}{(2k - \mathrm{i})(2k + \mathrm{i})}$$
$$= \frac{2k^2 - 7k\mathrm{i} - 4\mathrm{i}^2}{4k^2 - \mathrm{i}^2}$$
$$= \frac{2k^2 + 4 - 7k\mathrm{i}}{4k^2 + 1}$$
$$= \frac{2k^2 + 4}{4k^2 + 1} - \frac{7k\mathrm{i}}{4k^2 + 1}$$

Equating imaginary parts:

$$-\frac{7k}{4k^2 + 1} = \frac{7}{5}$$
$$-35k = 28k^2 + 7$$
$$28k^2 + 35k + 7 = 0$$
$$4k^2 + 5k + 1 = 0$$
$$(4k + 1)(k + 1) = 0$$

k is an integer, so

$k = -1$

b $z = \dfrac{-1 - 4\mathrm{i}}{-2 - \mathrm{i}} = \dfrac{1 + 4\mathrm{i}}{2 + \mathrm{i}}$

From part **a**:

$$\text{re}(z) = \frac{2k^2 + 4}{4k^2 + 1} = \frac{6}{5}$$

$$z = \frac{6}{5} + \frac{7}{5}\text{i}$$

$$\arg z = \tan^{-1}\left(\frac{7}{6}\right) = 0.862$$

14 **i** $w = a + b\text{i}$

$w* = a - b\text{i}$

$3(a + b\text{i}) + 2\text{i}(a - b\text{i}) = 17 + 8\text{i}$

Equating real parts:

$3a - 2b\text{i}^2 = 17$

$3a + 2b = 17$ ....[1]

Equating imaginary parts:

$3b + 2a = 8$ ....[2]

$2 \times$ [1] $6a + 4b = 34$ ....[3]

$3 \times$ [2] $6a + 9b = 24$ ....[4]

[4] − [3]:

$$5b = -10$$

$$b = -2$$

$$-6 + 2a = 8$$

$$a = 7$$

$w = 7 - 2\text{i}$

ii $\arg(z - 2\text{i}) = \frac{\pi}{6}$: Half line from $(0, 2)$, with angle $\frac{\pi}{6}$.

The equation of this half line:

$$\text{gradient} = \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$$

$$y = \frac{\sqrt{3}}{3}x + 2, x \geqslant 0$$

$|z - 3| = |z - 3\text{i}|$: Perpendicular bisector of the points $(0, 3)$ and $(3, 0)$.

This is the line with equation $y = x$.

The two lines intersect when:

$$x = \frac{\sqrt{3}}{3}x + 2$$

$$x\left(1 - \frac{\sqrt{3}}{3}\right) = 2$$

$$x = \frac{2}{\left(1 - \frac{\sqrt{3}}{3}\right)}$$

$$= \frac{6}{3 - \sqrt{3}}$$

$$y = \frac{6}{3 - \sqrt{3}}$$

So the complex number represented by P is:

$$z = \frac{6}{3 - \sqrt{3}} + \frac{6}{3 - \sqrt{3}}\text{i}$$

$$\arg\left(\frac{6}{3 - \sqrt{3}} + \frac{6}{3 - \sqrt{3}}\text{i}\right)$$

$$= \tan^{-1} 1 = \frac{\pi}{4}$$

$$\left|\frac{6}{3 - \sqrt{3}} + \frac{6}{3 - \sqrt{3}}\text{i}\right| = \sqrt{\left(\frac{6}{3 - \sqrt{3}}\right)^2 + \left(\frac{6}{3 - \sqrt{3}}\right)^2} = 6.69$$

$$z = 6.69\text{e}^{\frac{\text{i}\pi}{4}}$$

15 **i** $u + 2v = 2\text{i} \Rightarrow \text{i}u + 2\text{i}v = 2\text{i}^2$

$\text{i}u + 2\text{i}v = -2$ ....[1]

$\text{i}u + v = 3$ ....[2]

[2] – [1]:

$v(1 - 2\text{i}) = 5$

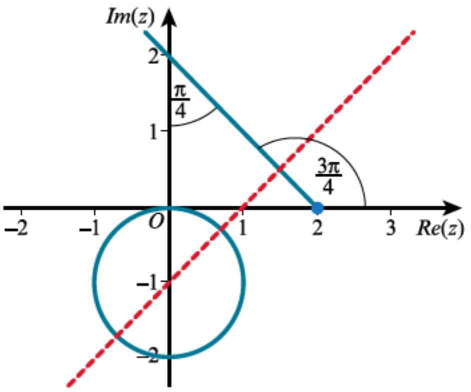
$$\begin{aligned} v &= \frac{5}{1-2\text{i}} \\ &= \frac{5(1+2\text{i})}{(1-2\text{i})(1+2\text{i})} \\ &= \frac{5+10\text{i}}{1-4\text{i}^2} \\ &= \frac{5+10\text{i}}{5} \\ &= 1+2\text{i} \end{aligned}$$

$$\begin{aligned} u &= 2\text{i} - 2v \\ &= 2\text{i} - 2 - 4\text{i} \\ &= -2 - 2\text{i} \end{aligned}$$

ii $|z + \text{i}| = 1$: Circle centre $(0, -1)$ radius 1.

Note that the real axis will be a tangent to this circle.

$\arg(w - 2) = \frac{3\pi}{4}$: Half line from the point $(2, 0)$ with angle $\frac{3\pi}{4}$.



$|z - w|$ is the distance between the points representing the complex numbers z and w.

The dotted line shows a direction that is perpendicular to the line but passes through the circle. The minimum distance between the half line and circle is the distance between the centre of the circle and the point where the dotted line meets the half line, less the radius of the circle.

$$\begin{aligned} &= 3\sin\left(\frac{\pi}{4}\right) - 1 \\ &= \frac{3\sqrt{2}}{2} - 1 \end{aligned}$$

16 **i**
$$\begin{aligned} w &= \frac{(1+\text{i})+\text{i}}{\text{i}(1+i)+2} \\ &= \frac{1+2\text{i}}{-1+2+\text{i}} \\ &= \frac{1+2\text{i}}{1+\text{i}} \\ &= \frac{(1+2\text{i})(1-\text{i})}{(1+\text{i})(1-\text{i})} \\ &= \frac{1+\text{i}-2\text{i}^2}{1-\text{i}^2} \\ &= \frac{3+\text{i}}{2} \\ &= \frac{3}{2} + \frac{1}{2}\text{i} \end{aligned}$$

ii $$z = \frac{z + \mathrm{i}}{\mathrm{i}z + 2}$$

$$\mathrm{i}z^2 + 2z = z + \mathrm{i}$$

$$\mathrm{i}z^2 + z - \mathrm{i} = 0$$

Let $z = x + \mathrm{i}y$

$$\mathrm{i}(x + \mathrm{i}y)^2 + x + \mathrm{i}y - \mathrm{i} = 0$$

$$\mathrm{i}(x^2 + 2xy\mathrm{i} + \mathrm{i}^2y^2) + x + \mathrm{i}y - \mathrm{i} = 0$$

$$\mathrm{i}x^2 + 2xy\mathrm{i}^2 - \mathrm{i}y^2 + x + \mathrm{i}y - \mathrm{i} = 0$$

Equating real parts:

$$-2xy + x = 0$$

$$x(1 - 2y) = 0$$

$$x = 0 \quad \text{or} \quad y = \frac{1}{2}$$

Reap part of z is negative, so $x \neq 0$.

So $y = \frac{1}{2}$.

Equating imaginary parts:

$$x^2 - y^2 + y - 1 = 0$$

$$x^2 - \left(\frac{1}{2}\right)^2 + \frac{1}{2} - 1 = 0$$

$$x^2 = \frac{3}{4}$$

$$x = \pm\frac{\sqrt{3}}{2}$$

$$x < 0$$

$$x = -\frac{\sqrt{3}}{2}$$

$$z = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$$

P3 **This exercise is for Pure Mathematics 3 students only.**

1 **a** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$$= \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} - \begin{pmatrix} -5 \\ 0 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 7 \\ -1 \end{pmatrix}$$

$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$

$$\mathbf{r} = \begin{pmatrix} -5 \\ 0 \\ 3 \end{pmatrix} - \lambda \begin{pmatrix} 6 \\ 7 \\ -1 \end{pmatrix}$$

b **i** The direction vectors of both lines will be perpendicular. This means that the scalar product of the two direction vectors will be zero.

Using the fact that the lines are perpendicular:

$$\begin{pmatrix} 6 \\ 7 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} m \\ 3 \\ 9 \end{pmatrix} = 0$$

$$6m + 21 - 9 = 0$$

$$6m = -12$$

$$m = -2$$

ii Lines intersect if:

$$\begin{pmatrix} -5 \\ 0 \\ 3 \end{pmatrix} + s\begin{pmatrix} 6 \\ 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \\ 9 \end{pmatrix}$$

$$-5 + 6s = 4 - 2t$$

$6s + 2t = 9$ .... [1]

$7s = 2 + 3t$

$7s - 3t = 2$ .... [2]

$3 - s = -3 + 9t$

$s + 9t = 6$ .... [3]

3 [1] $18s + 6t = 27$ ....[4]

2 [2] $14s - 6t = 4$ ....[5]

[4] + [5]:

$$32s = 31$$

$$s = \frac{31}{32}$$

Then from [2]:

$$7\left(\frac{31}{32}\right) - 3t = 2$$

$$\frac{217}{32} - \frac{64}{32} = 3t$$

$$3t = \frac{153}{32}$$

$$t = \frac{51}{32}$$

Trying in [3]:

$$s + 9t = \frac{31}{32} + \frac{459}{32} = \frac{490}{32} \neq 6$$

Equations are inconsistent, so the lines do not meet.

2

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 4$$

$$\frac{y}{y^2+4}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

$$\int \frac{y}{y^2+4}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x$$

$$\frac{1}{2}\ln\left|y^2+4\right| = \ln|x| + c$$

Using the fact that $y = 0$ when $x = 1$:

$$\frac{1}{2}\ln 4 = \ln 1 + c$$

$$c = \frac{1}{2}\ln 4 = \ln 2$$

$$\frac{1}{2}\ln\left|y^2+4\right| = \ln|x| + \ln 2$$

$$\ln\left|y^2+4\right| = 2\ln|x| + 2\ln 2$$

$$y^2 + 4 = \mathrm{e}^{2\ln|x|+\ln 4} = \mathrm{e}^{\ln 4}\mathrm{e}^{\ln x^2}$$

$$y^2 + 4 = 4x^2$$

$$y^2 = 4x^2 - 4$$

$$y^2 = 4\left(x^2 - 1\right)$$

3 a

> If a point lies on a line with a given equation then the position vector of the point can be written in the form of the line's equation.

$$\mathrm{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}$$

Using the fact that $P(p, q, -1)$ lies on the line:

$$2 + \lambda = -1$$

$$\lambda = -3$$

Position vector of P is:

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -25 \\ -1 \end{pmatrix}$$

$P(-2, -25, -1)$

$p = -2$

$q = -25$

b i

$$\overrightarrow{OQ} = \begin{pmatrix} -10 \\ 1 \\ -5 \end{pmatrix}$$

$$\left|\overrightarrow{OQ}\right| = \sqrt{(-10)^2 + 1^2 + (-5)^2}$$

$$= \sqrt{100 + 1 + 25}$$

$$= \sqrt{126}$$

Unit vector in direction of $\overrightarrow{OQ}$ is:

$$\frac{1}{\sqrt{126}}\begin{pmatrix}-10\\1\\-5\end{pmatrix}$$
$$=\frac{\sqrt{126}}{126}\begin{pmatrix}-10\\1\\-5\end{pmatrix}$$
$$=\frac{\sqrt{9}\sqrt{14}}{126}\begin{pmatrix}-10\\1\\-5\end{pmatrix}$$
$$=\frac{\sqrt{14}}{42}\begin{pmatrix}-10\\1\\-5\end{pmatrix}$$

ii

$$\overrightarrow{OP}=\begin{pmatrix}-2\\-25\\-1\end{pmatrix}$$
$$\overrightarrow{OQ}=\begin{pmatrix}-10\\1\\-5\end{pmatrix}$$

$$\begin{pmatrix}-2\\-25\\-1\end{pmatrix}\cdot\begin{pmatrix}-10\\1\\-5\end{pmatrix}=\sqrt{4+625+1}\sqrt{100+1+25}\cos\theta$$
$$\cos\theta=\frac{20-25+5}{\sqrt{4+625+1}\sqrt{100+1+25}}$$
$$=0$$

They are perpendicular.

$$\begin{aligned}\text{Triangle area} &= \frac{1}{2}\left|\overrightarrow{OP}\right|\left|\overrightarrow{OQ}\right|=\frac{1}{2}\sqrt{4+625+1}\sqrt{100+1+25}\\ &= \frac{1}{2}\sqrt{630}\sqrt{126}\\ &= \frac{1}{2}\sqrt{70}\sqrt{9}\sqrt{9}\sqrt{14}\\ &= \frac{9}{2}\sqrt{70}\sqrt{14}\\ &= \frac{9}{2}\sqrt{5}\sqrt{14}\sqrt{14}\\ &= 63\sqrt{5}\end{aligned}$$

4 **i**

$$z=\sqrt{3}+\mathrm{i}$$
$$|z|=\sqrt{\left(\sqrt{3}\right)^2+1^2}$$
$$=\sqrt{3+1}$$
$$=2$$
$$\arg z=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$$

If you are unsure when calculating arguments, draw an Argand diagram to help you.

ii

$$z=\sqrt{3}+\mathrm{i}$$
$$z^*=\sqrt{3}-\mathrm{i}$$

a $2z+z^*$

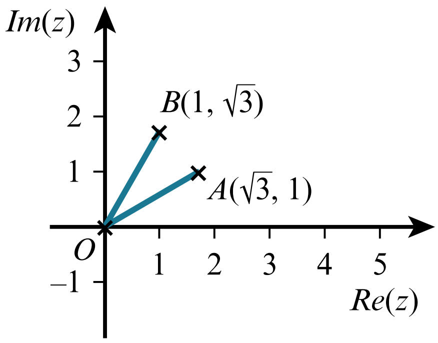
$$=2\left(\sqrt{3}+\mathrm{i}\right)+\sqrt{3}-\mathrm{i}$$
$$=2\sqrt{3}+2\mathrm{i}+\sqrt{3}-\mathrm{i}$$
$$=3\sqrt{3}+\mathrm{i}$$

b

Remember to make the denominator real by multiplying top and bottom by the complex conjugate of the denominator.

$$\frac{iz^*}{z}$$
$$= \frac{i\left(\sqrt{3}-i\right)}{\sqrt{3}+i}$$
$$= \frac{i\left(\sqrt{3}-i\right)\left(\sqrt{3}-i\right)}{\left(\sqrt{3}+i\right)\left(\sqrt{3}-i\right)}$$
$$= \frac{i\left(3-2\sqrt{3}i+i^2\right)}{3-i^2}$$
$$= \frac{3i-2\sqrt{3}i^2-i}{3+1}$$
$$= \frac{2\sqrt{3}+2i}{4}$$
$$= \frac{\sqrt{3}}{2}+\frac{1}{2}i$$

iii $iz^* = i\left(\sqrt{3}-i\right)$
$$= \sqrt{3}i - i^2$$
$$= 1+\sqrt{3}i$$



$$\arg z = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$
$$\arg iz^* = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$
$$\text{Angle } AOB = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

5
$$\sin 2\theta \frac{dx}{d\theta} = (x+1)\cos 2\theta$$
$$\frac{1}{x+1}\frac{dx}{d\theta} = \frac{\cos 2\theta}{\sin 2\theta}$$
$$\int \frac{1}{x+1}\frac{dx}{d\theta}\,d\theta = \int \frac{\cos 2\theta}{\sin 2\theta}\,d\theta$$
$$\ln|x+1| = \frac{1}{2}\ln|\sin 2\theta| + c$$

Using the fact that $\theta = \frac{\pi}{12}$ when $x = 0$:
$$\ln 1 = \frac{1}{2}\ln\left(\sin\frac{\pi}{6}\right) + c$$
$$c = -\frac{1}{2}\ln\frac{1}{2} = \frac{1}{2}\ln 2$$
$$\ln|x+1| = \frac{1}{2}\ln|\sin 2\theta| + \frac{1}{2}\ln 2$$

$\sin 2\theta > 0$ in the given domain.
$$\ln|x+1| = \frac{1}{2}\ln(2\sin 2\theta)$$
$$x+1 = \sqrt{2\sin 2\theta}$$
$$x = \sqrt{2\sin 2\theta} - 1$$

6 **a** $\overrightarrow{PS} = \overrightarrow{QR} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$

$$\begin{aligned}\overrightarrow{OS} &= \overrightarrow{OP} + \overrightarrow{PS} \\ &= \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \\ 10 \end{pmatrix}\end{aligned}$$

b $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -5 \end{pmatrix}$

$$\left|\overrightarrow{PQ}\right| = \sqrt{4+4+25} = \sqrt{33}$$

$$\overrightarrow{QR} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$$

$$\left|\overrightarrow{QR}\right| = \sqrt{1+9+25} = \sqrt{35}$$

c

$$\overrightarrow{QP} = \begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix}$$

$$\overrightarrow{QR} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix} = \sqrt{4+4+25}\sqrt{1+9+25}\cos\theta$$

$$\cos\theta = \frac{2-6+25}{\sqrt{33}\sqrt{35}}$$

$$\theta = 51.8^\circ$$

$$180^\circ - 51.8^\circ = 128.2^\circ$$

7 **i**

$$\begin{aligned}u &= \frac{(1+2\mathrm{i})^2}{2+\mathrm{i}} \\ &= \frac{1+4\mathrm{i}+4\mathrm{i}^2}{2+\mathrm{i}} \\ &= \frac{(1+4\mathrm{i}-4)(2-\mathrm{i})}{(2+\mathrm{i})(2-\mathrm{i})} \\ &= \frac{(-3+4\mathrm{i})(2-\mathrm{i})}{4-\mathrm{i}^2} \\ &= \frac{-6+3\mathrm{i}+8\mathrm{i}-4\mathrm{i}^2}{4+1} \\ &= \frac{-2+11\mathrm{i}}{5} \\ &= -\frac{2}{5}+\frac{11}{5}\mathrm{i}\end{aligned}$$

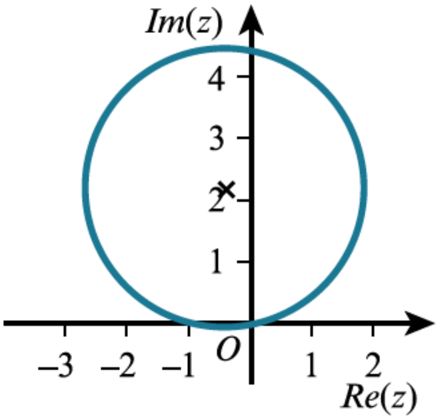
ii

$$\begin{aligned}|u| &= \sqrt{\left(-\frac{2}{5}\right)^2 + \left(\frac{11}{5}\right)^2} \\ &= \sqrt{\frac{4}{25}+\frac{121}{25}} \\ &= \sqrt{\frac{125}{25}} \\ &= \sqrt{5}\end{aligned}$$

$$|z-u| = \sqrt{5}$$

Circle, centre u and radius $\sqrt{5}$.

Note that because the radius of the circle is just a little more than the imaginary part of u, the circle dips below the real axis very slightly.



8 **i** $u = 2 + 2\text{i}$

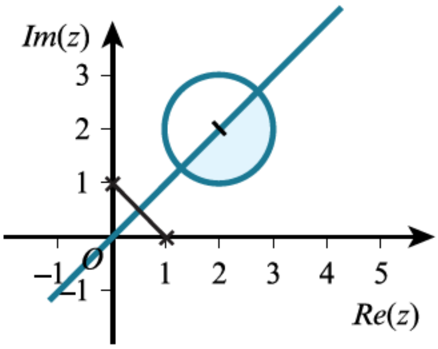
$|u| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$\arg u = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

ii $|z - 1| \leqslant |z - \text{i}|$ is the set of all points closer to the point $(1,\ 0)$ than the point $(0,\ 1)$, i.e. it is the region below the line $y = x$.

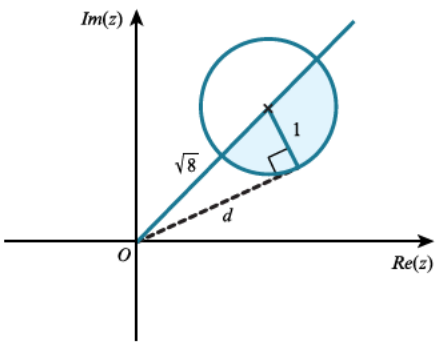
$|z - (2 + 2\text{i})| \leqslant 1$ is the inside of a circle, centre $(2,\ 2)$ and radius 1.

So shading the part of the circle that lies below the line $y = x$:



Note that the centre of this circle lies on the perpendicular bisector of $(1, 0)$ and $(0, 1)$.

iii



From the diagram it can be seen that the required modulus is d.

$d^2 = |u|^2 - 1^2 = 7$

$d = \sqrt{7}$

9 **i** $\frac{\text{d}x}{\text{d}t} = k(20 - x)$

Using the fact that $\frac{\text{d}x}{\text{d}t} = 1$ when $x = 0$:

$$1 = 20k$$
$$k = \frac{1}{20} = 0.05$$
$$\frac{dx}{dt} = 0.05(20 - x)$$

ii
$$\frac{dx}{dt} = 0.05(20 - x)$$
$$\frac{1}{20 - x}\frac{dx}{dt} = 0.05$$
$$\int \frac{1}{20 - x}\frac{dx}{dt}\,dt = \int 0.05\,dt$$
$$-\ln|20 - x| = 0.05t + c$$

Using the fact that $x = 0$ when $t = 0$:

$$-\ln 20 = c$$
$$-\ln|20 - x| = 0.05t - \ln 20$$
$$\ln|20 - x| = -0.05t + \ln 20$$
$$20 - x = e^{-0.05t}e^{\ln 20}$$
$$20 - x = 20e^{-0.05t}$$
$$x = 20 - 20e^{-0.05t}$$

iii When $t = 10$:

$$x = 20 - 20e^{-0.05(10)}$$
$$= 7.9 \text{ grams} \quad \text{(to 1 decimal place)}$$

iv $x = 20 - 20e^{-0.05t} \to 20 - 0 = 20.$

x gets closer and closer to 20.

10 **a** L_2 has the same direction vector as L_1:

$$\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

or

$$\mathbf{r} = -2\mathbf{i} + \mathbf{j} - \mathbf{k} + \mu(\mathbf{i} + 3\mathbf{k})$$

b Note that $\begin{pmatrix} -3 \\ k \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = -3 + 0 + 3 = 0$

Therefore, the vector $\begin{pmatrix} -3 \\ k \\ 1 \end{pmatrix}$ is perpendicular to the line L_1.

> You need to find the equation of a line through P, perpendicular to the line L_1, and find out where this line meets L_1. The shortest distance from P to L_1 will be the distance from P to this point of intersection.

Perpendicular line through P:

$$\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} + s\begin{pmatrix} -3 \\ k \\ 1 \end{pmatrix}$$

At point of intersection with L_1:

$$\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} + s\begin{pmatrix} -3 \\ k \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$-2 - 3s = -2 + t$

$3s + t = 0$ .... [1]

$1 + ks = 0$

$s = -\dfrac{1}{k}$ .... [2]

$-1 + s = 1 + 3t$

$s - 3t = 2$ .... [3]

Substituting [2] into [1] and then making consistent with [3] to ensure intersection:

$$-\frac{3}{k} + t = 0$$

$$t = \frac{3}{k}$$

Then from [3]:

$$-\frac{1}{k} - \frac{9}{k} = 2$$

$$-\frac{10}{k} = 2$$

$$k = -5$$

$$t = -\frac{3}{5}$$

$$s = \frac{1}{5}$$

Perpendicular vector of point of intersection is:

$$\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} + s\begin{pmatrix} -3 \\ k \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{13}{5} \\ 0 \\ -\frac{4}{5} \end{pmatrix}$$

Perpendicular vector from P to the line:

$$= \begin{pmatrix} -\frac{13}{5} \\ 0 \\ -\frac{4}{5} \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{5} \\ -1 \\ \frac{1}{5} \end{pmatrix}$$

$$\left|\begin{pmatrix} -\frac{3}{5} \\ -1 \\ \frac{1}{5} \end{pmatrix}\right| = \sqrt{\left(\frac{3}{5}\right)^2 + 1^2 + \left(\frac{1}{5}\right)^2}$$

$$= \sqrt{\frac{9}{25} + 1 + \frac{1}{25}}$$

$$= \sqrt{\frac{9 + 25 + 1}{25}}$$

$$= \sqrt{\frac{35}{25}}$$

$$= \sqrt{\frac{7}{5}}$$

11 **i**

$$\frac{dR}{dx} = R\left(\frac{1}{x} - 0.57\right)$$

$$\frac{1}{R}\frac{dR}{dx} = \frac{1}{x} - 0.57$$

$$\int \frac{1}{R}\frac{dR}{dx}\,dx = \int \left(\frac{1}{x} - 0.57\right) dx$$

$$\ln R = \ln x - 0.57x + c$$

Using the fact that $R = 16.8$ when $x = 0.5$:

$$\ln 16.8 = \ln 0.5 - 0.57(0.5) + c$$

$$c = \ln 33.6 + 0.285$$

$$\ln R = \ln x - 0.57x + \ln 33.6 + 0.285$$

$$\ln R = \ln x - 0.57x + 3.80$$

$$R = e^{\ln x - 0.57x + 3.80}$$

$$= xe^{3.80 - 0.57x}$$

ii

$$R = xe^{3.80-0.57x}$$

$$\frac{dR}{dx} = e^{3.80-0.57x} - 0.57xe^{3.80-0.57x}$$

At the maximum value, $\frac{dR}{dx}$:

$$e^{3.80-0.57x} - 0.57xe^{3.80-0.57x} = 0$$

$$e^{3.80-0.57x}(1 - 0.57x) = 0$$

$$x = \frac{1}{0.57}$$

$$R = \frac{1}{0.57}e^{3.80-1} = 28.850$$

12 **i**

$$z = \frac{9\sqrt{3} + 9i}{\sqrt{3} - i}$$

$$= \frac{(9\sqrt{3} + 9i)(\sqrt{3} + i)}{(\sqrt{3} - i)(\sqrt{3} + i)}$$

$$= \frac{27 + 9\sqrt{3}i + 9\sqrt{3}i + 9i^2}{3 - i^2}$$

$$= \frac{18 + 18\sqrt{3}i}{4}$$

$$= \frac{9}{2} + \frac{9\sqrt{3}}{2}i$$

$$|z| = \sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{9\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{81}{4} + \frac{243}{4}}$$

$$= \sqrt{\frac{324}{4}}$$

$$= \sqrt{81}$$

$$= 9$$

$$\arg z = \tan^{-1}\left(\frac{9\sqrt{3}}{9}\right) = \tan^{-1}\left(\sqrt{3}\right)$$

$$= \frac{\pi}{3}$$

$$z = re^{i\theta} = 9e^{\frac{i\pi}{3}}$$

ii

> Remember that you can add or subtract 2π from the argument of any complex number without changing it.

Square roots are:

$$\left(9e^{\frac{i\pi}{3}}\right)^{\frac{1}{2}} \quad \text{and} \quad \left(9e^{\frac{i\pi}{3} - 2\pi}\right)^{\frac{1}{2}}$$

$= 3e^{\frac{i\pi}{6}}$ and $3e^{-\frac{5i\pi}{6}}$

13

$$\frac{dy}{dx} = \frac{6ye^{3x}}{2+e^{3x}}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{6e^{3x}}{2+e^{3x}}$$

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{6e^{3x}}{2+e^{3x}}\,dx$$

$$\ln|y| = 2\ln|2+e^{3x}| + c$$

Using the fact that $y = 36$ when $x = 0$:

$$\ln 36 = 2\ln(2+1) + c$$

$$c = \ln 36 - 2\ln 3 = \ln 4$$

$$\ln|y| = 2\ln|2+e^{3x}| + \ln 4$$

$$\ln|y| = \ln\left(2+e^{3x}\right)^2 + \ln 4$$

$$\ln|y| = \ln 4\left(2+e^{3x}\right)^2$$

$$y = 4\left(2+e^{3x}\right)^2$$

14 **a** $\overrightarrow{AB} = \begin{pmatrix} 7 \\ 1 \\ 27 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ 20 \end{pmatrix}$

Line AB has equation:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 7 \\ -1 \\ 20 \end{pmatrix}$$

Lines intersect when:

$$\begin{pmatrix} 0 \\ 2 \\ 7 \end{pmatrix} + s\begin{pmatrix} 7 \\ -1 \\ 20 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -10 \end{pmatrix} + t\begin{pmatrix} 8 \\ -1 \\ 3 \end{pmatrix}$$

$7s = 1 + 8t$

$7s - 8t = 1$ ....[1]

$2 - s = 2 - t$

$s - t = 0$

$s = t$ ....[2]

$7 + 20s = -10 + 3t$

$20s - 3t = -17$ ....[3]

Substituting [2] into [1]:

$7s - 8s = -s = 1$

$s = -1 = t$

Checking in [3]:

$-20 - 3(-1) = -17$

Position vector is $\begin{pmatrix} 0 \\ 2 \\ 7 \end{pmatrix} - \begin{pmatrix} 7 \\ -1 \\ 20 \end{pmatrix} = \begin{pmatrix} -7 \\ 3 \\ -13 \end{pmatrix}$

b $\begin{pmatrix} 7 \\ -1 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ -1 \\ 3 \end{pmatrix} = \sqrt{49+1+400}\sqrt{64+1+9}\cos\theta$

$$\cos\theta = \frac{56+1+60}{\sqrt{450}\sqrt{74}}$$

$$\theta = 50.1^\circ$$

If you get an obtuse angle, which will happen if the scalar product is negative, subtract your answer from 180° to get the acute angle that you need.

15

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y\left(1 - 2x^2\right)$$

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-2x^2}{x}$$

$$\int \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{1-2x^2}{x}\,\mathrm{d}x$$

$$\ln|y| = \int\left(\frac{1}{x} - 2x\right)\mathrm{d}x$$

$$\ln|y| = \ln|x| - x^2 + c$$

Using the fact that $y = 2$ when $x = 1$:

$$\ln 2 = \ln 1 - 1 + c$$

$$c = 1 + \ln 2$$

$$\ln|y| = \ln|x| - x^2 + 1 + \ln 2$$

$$\ln|y| = \ln 2|x| - x^2 + 1$$

$$y = \mathrm{e}^{\ln 2x}\,\mathrm{e}^{1-x^2}$$

$$y = 2x\mathrm{e}^{1-x^2}$$

16 **i** $|u| = \frac{|6-3\mathrm{i}|}{|1+2\mathrm{i}|} = \frac{\sqrt{36+9}}{\sqrt{1+4}}$

$$= \frac{\sqrt{45}}{\sqrt{5}}$$

$$= \sqrt{9}$$

$$= 3$$

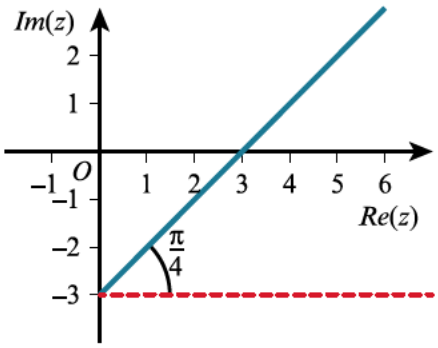
$$u = \frac{(6-3\mathrm{i})(1-2\mathrm{i})}{(1+2\mathrm{i})(1-2\mathrm{i})}$$

$$= \frac{6 - 12\mathrm{i} - 3\mathrm{i} + 6\mathrm{i}^2}{1 - 4\mathrm{i}^2}$$

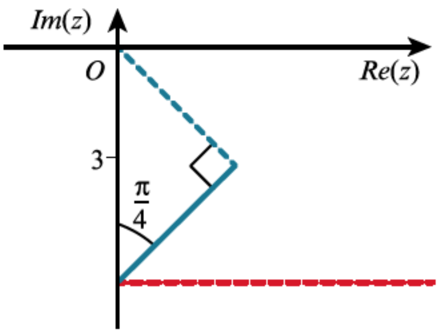
$$= \frac{-15\mathrm{i}}{5} = -3\mathrm{i}$$

$$\arg(u) = -\frac{\pi}{2}$$

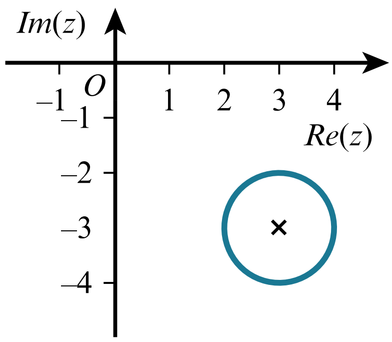
ii $\arg(z-u) = \frac{\pi}{4}$ represents the half-line from $(0, -3)$.



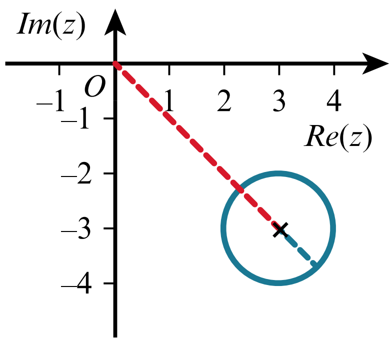
Least $|z| = 3\sin\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$



iii $|z - (1+\mathrm{i})\,u| = 1$ represents the circle, centre $3 - 3\mathrm{i}$, radius 1.



Greatest $|z| = \sqrt{3^2 + (-3)^2} + 1 = 3\sqrt{2} + 1$



17 **a** $\overrightarrow{AB} = \begin{pmatrix} 3 \\ m \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ m-1 \\ -1 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 1 \\ -5 \\ m+11 \end{pmatrix} - \begin{pmatrix} 3 \\ m \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -5-m \\ m+7 \end{pmatrix}$

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$$

$$\begin{pmatrix} 1 \\ m-1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -5-m \\ m+7 \end{pmatrix} = 0$$

$$-2 + (m-1)(-5-m) - 1(m+7) = 0$$

$$-2 - 5m - m^2 + 5 + m - m - 7 = 0$$

$$m^2 + 5m + 4 = 0$$

$$(m+1)(m+4) = 0$$

$$m = -1, -4$$

b $\begin{pmatrix} 2 \\ -4 \\ -2m \end{pmatrix} = \frac{1}{2}\left\{\begin{pmatrix} 3 \\ m \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ -5 \\ m+11 \end{pmatrix}\right\}$

$$-4 = \frac{1}{2}m - \frac{5}{2}$$

$$-8 = m - 5$$

$$m = -3$$

c $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}$

Line AB:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + s\begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}$$

$\overrightarrow{CD} = \begin{pmatrix} 2 \\ -4 \\ 10 \end{pmatrix} - \begin{pmatrix} 1 \\ -5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$

Line CD:

$$\mathbf{r} = \begin{pmatrix} 1 \\ -5 \\ 6 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

Intersect when:

$$\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + s\begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 6 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$2 + s = 1 + t$

$s - t = -1$ ....[1]

$1 - 6s = -5 + t$

$6s + t = 6$ ....[2]

$5 - s = 6 + 4t$

$s + 4t = -1$ ....[3]

[1] + [2]:

$7s = 5$

$s = \frac{5}{7}$

Then from [1]:

$\frac{5}{7} - t = -1$

$t = \frac{12}{7}$

Trying in [3]:

$s + 4t = \frac{5}{7} + \frac{48}{7} = \frac{53}{7} \neq -1$

Equations are inconsistent, so the lines do not intersect.

18 **i** $w = \frac{22 + 4\mathrm{i}}{(2 - \mathrm{i})^2}$

$= \frac{(22 + 4\mathrm{i})(2 + \mathrm{i})^2}{(2 - \mathrm{i})^2(2 + \mathrm{i})^2}$

$= \frac{(22 + 4\mathrm{i})\left(4 + 4\mathrm{i} + \mathrm{i}^2\right)}{\left(4 - \mathrm{i}^2\right)^2}$

$= \frac{(22 + 4\mathrm{i})(3 + 4\mathrm{i})}{5^2}$

$= \frac{66 + 88\mathrm{i} + 12\mathrm{i} + 16\mathrm{i}^2}{25}$

$= \frac{50 + 100\mathrm{i}}{25}$

$= 2 + 4\mathrm{i}$

ii $w + p = (2 + p) + 4\mathrm{i}$

$\arg(w + p) = \tan^{-1}\left(\frac{4}{2 + p}\right) = \frac{3\pi}{4}$

$\frac{4}{2 + p} = -1$

$4 = -2 - p$

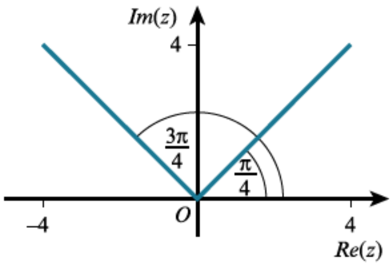
$p = -6$

$\tan^{-1}\left(\frac{4}{2 + p}\right) = \frac{\pi}{4}$

$\frac{4}{2 + p} = 1$

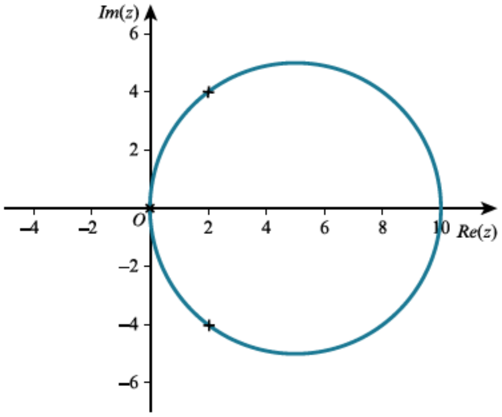
$p = 2$

$-6 \leqslant p \leqslant 2$



iii $w = 2 + 4\mathrm{i}$

$w^* = 2 - 4\mathrm{i}$



Note that the centre must lie on the real axis, at $(a, 0)$, say. The distance from the origin to $(a, 0)$ must be the same as the distance from S to $(a, 0)$, because both the origin and S lie on the circle, the same distance from the centre.

$|a| = |w - a|$

$a^2 = |w - a|^2$

$a^2 = |(2 - a) + 4\mathrm{i}|^2$

$a^2 = (2 - a)^2 + 16$

$a^2 = a^2 - 4a + 4 + 16$

$4a = 20$

$a = 5$

Radius $= a = 5$

Centre is $(5, 0)$.

Circle has equation $|z - 5| = 5$

19 You will notice the need to factorise a denominator and use partial fractions in worked solution **19**.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x\left(3y^2 + 10y + 3\right)$$

$$\frac{1}{(3y+1)(y+3)}\frac{\mathrm{d}y}{\mathrm{d}x} = 4x$$

$$\int \frac{1}{(3y+1)(y+3)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int 4x\,\mathrm{d}x$$

$$\frac{1}{(3y+1)(y+3)} = \frac{A}{3y+1} + \frac{B}{y+3}$$

$$1 = A(y+3) + B(3y+1)$$

Letting $y = -3$:

$1 = 0 - 8B$

$B = -\frac{1}{8}$

Letting $y = -\frac{1}{3}$:

$$1 = \frac{8}{3}A$$
$$A = \frac{3}{8}$$
$$\int \frac{3}{8(3y+1)} - \frac{1}{8(y+3)}\,\mathrm{d}y = \int 4x\,\mathrm{d}x$$
$$\frac{1}{8}\ln|3y+1| - \frac{1}{8}\ln|y+3| = 2x^2 + c$$

Using the fact that $y = 1$ when $x = 0$:

$$\frac{1}{8}\ln 4 - \frac{1}{8}\ln 4 = 0 + c$$
$$c = 0$$
$$\frac{1}{8}\ln|3y+1| - \frac{1}{8}\ln|y+3| = 2x^2$$
$$\frac{1}{8}\ln\left|\frac{3y+1}{y+3}\right| = 2x^2$$
$$\ln\left|\frac{3y+1}{y+3}\right| = 16x^2$$
$$\frac{3y+1}{y+3} = \mathrm{e}^{16x^2}$$
$$3y+1 = y\mathrm{e}^{16x^2} + 3\mathrm{e}^{16x^2}$$
$$y\left(3 - \mathrm{e}^{16x^2}\right) = 3\mathrm{e}^{16x^2} - 1$$
$$y = \frac{3\mathrm{e}^{16x^2} - 1}{3 - \mathrm{e}^{16x^2}}$$

20 **i** $\frac{\mathrm{d}V}{\mathrm{d}t} = -k\sqrt{h} \qquad k > 0$

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(h\tan 60^\circ)^2 h$$
$$= \frac{1}{3}\pi\left(h\sqrt{3}\right)^2 h$$
$$= \pi h^3$$

By the chain rule:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t}$$
$$-k\sqrt{h} = 3\pi h^2 \frac{\mathrm{d}h}{\mathrm{d}t}$$
$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{-k\sqrt{h}}{3\pi h^2} = -Ah^{-\frac{3}{2}} \text{ where } A = \frac{k}{3\pi} > 0$$

ii

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -Ah^{-\frac{3}{2}}$$
$$h^{\frac{3}{2}}\frac{\mathrm{d}h}{\mathrm{d}t} = -A$$
$$\int h^{\frac{3}{2}}\frac{\mathrm{d}h}{\mathrm{d}t}\,\mathrm{d}t = \int -A\,\mathrm{d}t$$
$$\frac{2}{5}h^{\frac{5}{2}} = -At + c$$

Using the fact that $h = H$ when $t = 0$:

$$\frac{2}{5}H^{\frac{5}{2}} = c$$
$$\frac{2}{5}h^{\frac{5}{2}} = -At + \frac{2}{5}H^{\frac{5}{2}}$$

Using the fact that $h = 0$ when $t = 60$:

$$0 = -60A + \frac{2}{5}H^{\frac{5}{2}}$$
$$A = \frac{1}{150}H^{\frac{5}{2}}$$

$$\frac{2}{5}h^{\frac{5}{2}} = -\frac{1}{150}H^{\frac{5}{2}}t + \frac{2}{5}H^{\frac{5}{2}}$$
$$t = \frac{60H^{\frac{5}{2}} - 60h^{\frac{5}{2}}}{H^{\frac{5}{2}}}$$

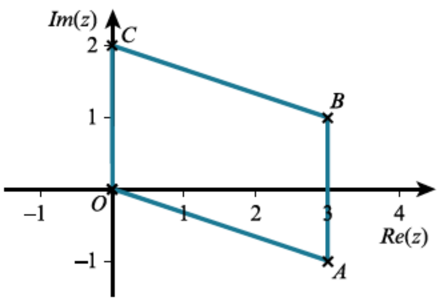
iii When $h = \frac{1}{2}H$:

$$t = \frac{60H^{\frac{5}{2}} - 60\left(\frac{1}{2}H\right)^{\frac{5}{2}}}{H^{\frac{5}{2}}}$$
$$= \frac{60 - 60\left(\frac{1}{2}\right)^{\frac{5}{2}}}{1}$$
$$= 49.4 \text{ (to 1 decimal place)}$$

21 **a** $\overrightarrow{AB} = \begin{pmatrix} 9 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$
$$= \begin{pmatrix} 7 \\ -6 \\ -9 \end{pmatrix}$$
$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$$
$$\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 7 \\ -6 \\ -9 \end{pmatrix}$$

b $\begin{pmatrix} 7 \\ -6 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \sqrt{49 + 36 + 81}\sqrt{1 + 9 + 4}\cos\theta$
$$\cos\theta = \frac{7 - 18 - 18}{\sqrt{166}\sqrt{14}}$$
$$\theta = 127.0^\circ$$

22 **i** $u = 3 - \mathrm{i}$
$$u* = 3 + \mathrm{i}$$
$$u* - u = 3 + \mathrm{i} - (3 - \mathrm{i})$$
$$= 2\mathrm{i}$$



The quadrilateral is a parallelogram.

ii $\frac{u*}{u} = \frac{3 + \mathrm{i}}{3 - \mathrm{i}}$
$$= \frac{(3 + \mathrm{i})(3 + \mathrm{i})}{(3 - \mathrm{i})(3 + \mathrm{i})}$$
$$= \frac{9 + 6\mathrm{i} + \mathrm{i}^2}{9 - \mathrm{i}^2}$$
$$= \frac{8 + 6\mathrm{i}}{9 + 1}$$
$$= \frac{4}{5} + \frac{3}{5}\mathrm{i}$$

iii $\arg\left(\frac{4}{5}+\frac{3}{5}\mathrm{i}\right)=\tan^{-1}\left(\frac{\left(\frac{3}{5}\right)}{\left(\frac{4}{5}\right)}\right)=\tan^{-1}\left(\frac{3}{4}\right)$

But

$$\arg\left(\frac{4}{5}+\frac{3}{5}\mathrm{i}\right)=\arg\left(\frac{3+\mathrm{i}}{3-\mathrm{i}}\right)=\arg(3+\mathrm{i})-\arg(3-\mathrm{i})$$

$$\tan^{-1}\left(\frac{3}{4}\right)=\tan^{-1}\left(\frac{1}{3}\right)-\tan^{-1}\left(-\frac{1}{3}\right)$$

$$\tan^{-1}\left(\frac{3}{4}\right)=2\tan^{-1}\left(\frac{1}{3}\right)$$

because $-\tan^{-1}\left(-\frac{1}{3}\right)=2\tan^{-1}\left(\frac{1}{3}\right)$

23 **i** $u^4=\left(1+\left(\sqrt{2}\right)\mathrm{i}\right)^4$

$$=1+4\sqrt{2}\mathrm{i}+6\left(\sqrt{2}\mathrm{i}\right)^2+4\left(\sqrt{2}\mathrm{i}\right)^3+\left(\sqrt{2}\mathrm{i}\right)^4$$

$$=1+4\sqrt{2}\mathrm{i}+12\mathrm{i}^2+8\sqrt{2}\mathrm{i}^3+4\mathrm{i}^4$$

$$=1-12+4+(4\sqrt{2}-8\sqrt{2})\mathrm{i}$$

$$=-7-4\sqrt{2}\mathrm{i}$$

$$u^2=\left(1+\left(\sqrt{2}\right)\mathrm{i}\right)^2=1+2\sqrt{2}\mathrm{i}+\left(\sqrt{2}\mathrm{i}\right)^2$$

$$=1+2\sqrt{2}\mathrm{i}-2$$

$$=-1+2\sqrt{2}\mathrm{i}$$

$$u^4+u^2+2u+6$$

$$=-7-4\sqrt{2}\mathrm{i}-1+2\sqrt{2}\mathrm{i}+2\left(1+\sqrt{2}\mathrm{i}\right)+6$$

$$=-7-1+2+6+(-4+2+2)\sqrt{2}\mathrm{i}$$

$$=0$$

u is a root as required.

Second complex root is $u^*=1-\left(\sqrt{2}\right)\mathrm{i}$.

ii $z-\left(1+\sqrt{2}\mathrm{i}\right)$ and $z-\left(1-\sqrt{2}\mathrm{i}\right)$ are factors of $\mathrm{p}(x)$ by the factor theorem.

$\left(z-1-\sqrt{2}\mathrm{i}\right)\left(z-1+\sqrt{2}\mathrm{i}\right)$ is a factor of $\mathrm{p}(x)$.

$z^2-z+z\sqrt{2}\mathrm{i}-z+1-\sqrt{2}\mathrm{i}-\sqrt{2}z\mathrm{i}+\sqrt{2}\mathrm{i}-2\mathrm{i}^2$ is a factor of $\mathrm{p}(x)$.

z^2-2z+3 is a factor of $\mathrm{p}(x)$

Dividing:

$$\begin{array}{r} z^2+2z+2 \\ z^2-2z+3\overline{\big)\,z^4+0z^3+z^2+2z+6} \\ \underline{z^4-2z^3+3z^2} \\ 2z^3-2z^2+2z \\ \underline{2z^3-4z^2+6z} \\ 2z^2-4z+6 \\ \underline{2z^2-4z+6} \\ 0 \end{array}$$

$z^4+z^2+2z+6=\left(z^2-2z+3\right)\left(z^2+2z+2\right)=0$

$z=1\pm\sqrt{2}\mathrm{i}$

or

$$z^2+2z+2=0$$

$$(z+1)^2-1+2=0$$

$$(z+1)^2=-1$$

$$z+1=\pm\mathrm{i}$$

$$z=-1\pm\mathrm{i}$$

24 **a**

The vector **k** is a unit vector with length 1 cm. Three of these are required to reach N from O.

$h = 3$

b $\overrightarrow{ON} = \mathbf{j} + 3\mathbf{k}$

$\left|\overrightarrow{ON}\right| = \sqrt{1+9} = \sqrt{10}$

Unit vector in direction of $\overrightarrow{ON}$

$= \frac{1}{\sqrt{10}}(\mathbf{j} + 3\mathbf{k})$

c

$$\begin{aligned}\overrightarrow{MN} &= -5\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}\\ \overrightarrow{ME} &= 5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}\\ (-5\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) \cdot (5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) &= \sqrt{25+9+9}\sqrt{25+16+9}\cos\theta\\ \cos\theta &= \frac{-25+12+9}{\sqrt{43}\sqrt{50}}\\ \theta &= 94.9^\circ\end{aligned}$$

d $\overrightarrow{MF} = 5\mathbf{i} + 3\mathbf{k}$

$\mathbf{r} = \overrightarrow{OM} + \lambda\overrightarrow{MF}$

$\mathbf{r} = 5\mathbf{i} + 4\mathbf{j} + \lambda(5\mathbf{i} + 3\mathbf{k})$

25

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x\mathrm{e}^{3x}}{y^2}$$

$$y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 6x\mathrm{e}^{3x}$$

$$\int y^2\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int 6x\mathrm{e}^{3x}\,\mathrm{d}x$$

Using integration by parts:

$$\frac{1}{3}y^3 = \frac{1}{3}\left(6x\mathrm{e}^{3x}\right) - \int\frac{1}{3}(6)\,\mathrm{e}^{3x}\,\mathrm{d}x$$

$$\frac{1}{3}y^3 = 2x\mathrm{e}^{3x} - \int 2\mathrm{e}^{3x}\,\mathrm{d}x$$

$$\frac{1}{3}y^3 = 2x\mathrm{e}^{3x} - \frac{2}{3}\mathrm{e}^{3x} + c$$

$$y^3 = 6x\mathrm{e}^{3x} - 2\mathrm{e}^{3x} + c'$$

Using the fact that $y = 2$ when $x = 0$:

$$8 = 0 - 2 + c'$$

$$c' = 10$$

$$y^3 = 6x\mathrm{e}^{3x} - 2\mathrm{e}^{3x} + 10$$

When $x = 0.5$:

$$y = \sqrt[3]{6\,(0.5)\,\mathrm{e}^{3(0.5)} - 2\mathrm{e}^{3(0.5)} + 10} = 2.44 \text{ (to 2 decimal places)}$$

26 **i**

$$(x+\mathrm{i}y)^2 = 7 - 6\sqrt{2}\mathrm{i}$$

$$x^2 + 2xy\mathrm{i} + y^2\mathrm{i}^2 = 7 - 6\sqrt{2}\mathrm{i}$$

Equating real and imaginary parts:

$x^2 - y^2 = 7$ ....[1]

$2xy = -6\sqrt{2}$

$xy = -3\sqrt{2}$

$y = -\frac{3\sqrt{2}}{x}$ ....[2]

Substituting [2] into [1]:

$$x^2 - \left(-\frac{3\sqrt{2}}{x}\right)^2 = 7$$

$$x^2 - \frac{18}{x^2} = 7$$

$$\left(x^2\right)^2 - 18 = 7x^2$$

$$\left(x^2\right)^2 - 7x^2 - 18 = 0$$

$$\left(x^2 - 9\right)\left(x^2 + 2\right) = 0$$

$x^2 = 9$ or $x^2 = -2$

x is real.

$x = 3$ or -3

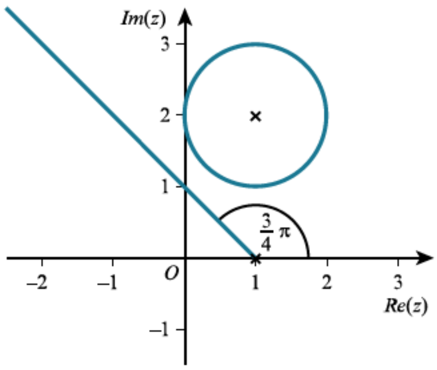
$y = -\sqrt{2}$ or $\sqrt{2}$

Square roots are:

$3 - \sqrt{2}\text{i}$ and $-3 + \sqrt{2}\text{i}$

ii **a** $|w - (1 + 2\text{i})| = 1$: circle, centre $(1, 2)$, radius 1.

$\arg(z - 1) = \dfrac{3}{4}\pi$: half-line, above and to the left of the point $(1, 0)$.



b This is the shortest distance between the two loci. Given that the half line has gradient 1 and the radius of the circle is 1, the diagram shows that this shortest distance is:

[Distance from $(0, 1)$ to $(1, 2)$] – [radius of the circle]

$= \sqrt{1^2 + 1^2} - 1$
$= \sqrt{2} - 1$

27 **a** If the lines intersect:

$$\begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$-2 + s = -4 - t$

$s + t = -2$ .... [1]

$3 - s = 5$

$s = -2$ .... [2]

$2s = 2 + t$

$2s - t = 2$ .... [3]

Substituting [2] into [1]:

$t = 0$

Checking equation [3]:

$2s - t = -4 \neq 2$

The equations are inconsistent, so the lines do not intersect.

b

> Notice that you have done almost all of the work for this part in part **a**. You now just need to make sure that the third equation matches.

$-2 + s = -4 - t$

$s + t = -2$ ....[1]

$3 - s = 5$

$s = -2$ ....[2]

$2s = m + t$

$2s - t = m$ ....[3]

Substituting [2] into [1]:

$t = 0$

These values must also work in equation [3].

$$2s - t = -4 = m$$
$$m = -4$$

c $$\begin{pmatrix}1\\-1\\2\end{pmatrix} \cdot \begin{pmatrix}-1\\0\\1\end{pmatrix} = \sqrt{1+1+4}\sqrt{1+1}\cos\theta$$
$$\cos\theta = \frac{-1+0+2}{\sqrt{6}\sqrt{2}}$$
$$\theta = 73.2^\circ$$

28 **i** $$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(1800-N)}{3600}$$
$$\frac{3600}{N(1800-N)}\frac{\mathrm{d}N}{\mathrm{d}t} = 1$$
$$\int \frac{3600}{N(1800-N)}\frac{\mathrm{d}N}{\mathrm{d}t}\,\mathrm{d}t = \int 1\,\mathrm{d}t$$
$$\frac{3600}{N(1800-N)} = \frac{A}{N} + \frac{B}{1800-N}$$
$$3600 = A(1800-N) + BN$$

Letting $N = 0$:

$$3600 = 1800A$$
$$A = 2$$

Letting $N = 1800$:

$$3600 = 0 + 1800B$$
$$B = 2$$
$$\frac{3600}{N(1800-N)} = \frac{2}{N} + \frac{2}{1800-N}$$
$$\int \left(\frac{2}{N} + \frac{2}{1800-N}\right)\frac{\mathrm{d}N}{\mathrm{d}t}\,\mathrm{d}t = \int 1\,\mathrm{d}t$$
$$2\ln|N| - 2\ln|1800-N| = t + c$$
$$2\ln\left|\frac{N}{1800-N}\right| = t + c$$

Using the fact that $N = 300$ when $t = 0$:

$$2\ln\frac{1}{5} = 0 + c$$
$$c = 2\ln\frac{1}{5}$$
$$2\ln\left|\frac{N}{1800-N}\right| = t + 2\ln\frac{1}{5}$$
$$2\ln\left|\frac{N}{1800-N}\right| - 2\ln\frac{1}{5} = t$$
$$2\ln\left|\frac{5N}{1800-N}\right| = t$$
$$\ln\left|\frac{5N}{1800-N}\right| = \frac{t}{2}$$
$$\frac{5N}{1800-N} = \mathrm{e}^{\frac{t}{2}}$$
$$5N = 1800\mathrm{e}^{\frac{t}{2}} - N\mathrm{e}^{\frac{t}{2}}$$
$$\left(5 + \mathrm{e}^{\frac{t}{2}}\right)N = 1800\mathrm{e}^{\frac{t}{2}}$$
$$N = \frac{1800\mathrm{e}^{\frac{t}{2}}}{5 + \mathrm{e}^{\frac{t}{2}}}$$

ii $$\frac{1800\mathrm{e}^{\frac{t}{2}}}{5 + \mathrm{e}^{\frac{t}{2}}} \to 1800 \text{ as } t \to \infty$$

29 **i** $(x+\mathrm{i}y)^2 = 1 - 2\sqrt{6}\mathrm{i}$

$x^2 + 2xy\mathrm{i} + y^2\mathrm{i}^2 = 1 - 2\sqrt{6}\mathrm{i}$

Equating real and imaginary parts:

$x^2 - y^2 = 1$ .... [1]

$2xy = -2\sqrt{6}$

$xy = -\sqrt{6}$

$y = -\dfrac{\sqrt{6}}{x}$ .... [2]

Substituting [2] into [1]:

$$x^2 - \left(-\frac{\sqrt{6}}{x}\right)^2 = 1$$

$$x^2 - \frac{6}{x^2} = 1$$

$$\left(x^2\right)^2 - 6 = x^2$$

$$\left(x^2\right)^2 - x^2 - 6 = 0$$

$$\left(x^2 - 3\right)\left(x^2 + 2\right) = 0$$

$x^2 = 3$ or $x^2 = -2$

x is real.

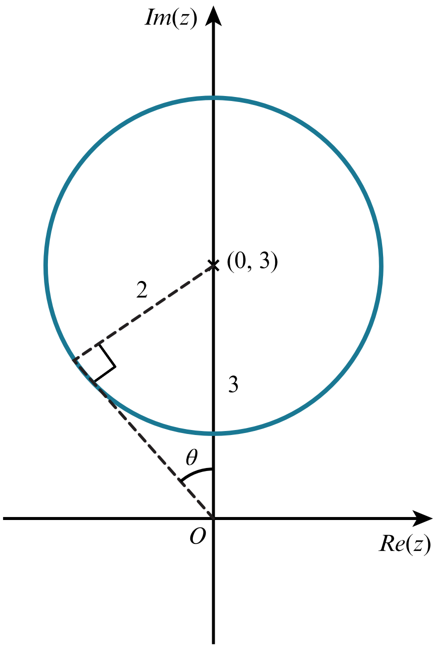
$x = \sqrt{3}$ or $-\sqrt{3}$

$y = -\sqrt{2}$ or $\sqrt{2}$

Square roots are:

$\sqrt{3} - \sqrt{2}\mathrm{i}$ and $-\sqrt{3} + \sqrt{2}\mathrm{i}$

ii This is the inside of the circle, radius 2, centre (0, 3).



Greatest argument is:

$\theta + \dfrac{\pi}{2}$

$= \sin^{-1}\left(\dfrac{2}{3}\right) + \dfrac{\pi}{2}$

$= 2.30$

30 **i** $\dfrac{\mathrm{d}V}{\mathrm{d}t} = $ (flow in) – (flow out)

$$\frac{dV}{dt} = 80 - kV$$
$$\frac{1}{80-kV}\frac{dV}{dt} = 1$$
$$\int \frac{1}{80-kV}\frac{dV}{dt}\,dt = \int 1\,dt$$
$$-\frac{1}{k}\ln|80-kV| = t + c$$
$$\ln|80-kV| = -kt + c'$$

Using the fact the $V = 0$ when $t = 0$:

$$\ln 80 = c'$$
$$\ln|80-kV| = -kt + \ln 80$$
$$80 - kV = e^{-kt}e^{\ln 80}$$
$$80 - kV = 80e^{-kt}$$
$$kV = 80 - 80e^{-kt}$$
$$V = \frac{1}{k}\left(80 - 80e^{-kt}\right)$$

ii Using $k_{n+1} = \dfrac{4 - 4e^{-15k_n}}{25}$ with $k_0 = 0.1$:

$k_1 = 0.1243$
$k_2 = 0.1352$
$k_3 = 0.1389$
$k_4 = 0.1401$
$k_5 = 0.1404$
$k_6 = 0.1405$
$k_7 = 0.1406$
$k_8 = 0.1406$
$k_9 = 0.1406$
$k = 0.14$ (to 2 significant figures)

iii $V = \dfrac{1}{0.14}\left(80 - 80e^{-0.14t}\right)$

When $t = 20$:

$V = \dfrac{1}{0.14}\left(80 - 80e^{-0.14(20)}\right) = 540\text{ cm}^3$

$e^{-0.14t} \to 0$ as $t \to \infty$

$V \to \frac{80}{0.14} = 571\,\text{cm}^3$

PURE MATHEMATICS 2 PRACTICE EXAM-STYLE PAPER

All worked solutions within this resource have been written by the author. In examinations, the way marks are awarded may be different.

1 Using the quotient rule:

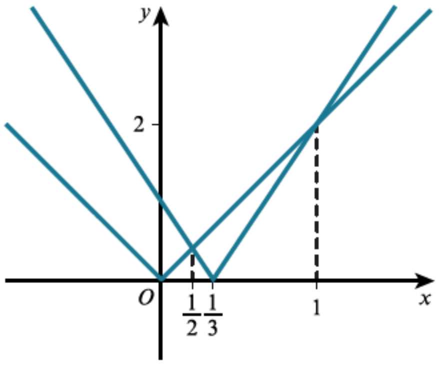
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos x}{\sin x}\right)$$
$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$
$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}$$
$$= -\frac{1}{\sin^2 x}$$
$$= -\operatorname{cosec}^2 x$$

2 $|3x - 1| = |2x|$

$3x - 1 = 2x$ or $3x - 1 = -2x$

$x = 1$ $\quad$ $5x = 1$

$x = \frac{1}{5}$



$x \geqslant 1$ or $x \leqslant \frac{1}{5}$

3
$$3^{2x} - 3^{x+1} = 10$$
$$(3^x)^2 - 3(3^x) - 10 = 0$$
$$(3^x - 5)(3^x + 2) = 0$$

$3^x = 5$ or $3^x = -2$

$$3^x > 0$$
$$3^x = 5$$
$$\ln 3^x = \ln 5$$
$$x \ln 3 = \ln 5$$

$x = \frac{\ln 5}{\ln 3} = 1.46$ to 3 significant figures

4 **a** $\mathrm{p}(x) = x^3 + 8x^2 + px - 25$

By the remainder theorem:

$$\mathrm{p}(1) = R$$
$$1 + 8 + p - 25 = R$$

$p - 16 = R$ ....[1]

Again, by the remainder theorem:

$$p(-2) = -R$$
$$-8 + 32 - 2p - 25 = -R$$
$$-1 - 2p = -R$$

$1 + 2p = R$ ....[2]

[1] = [2]

$$p - 16 = 1 + 2p$$
$$p = -17$$

b $\mathrm{p}(x) = x^3 + 8x^2 - 17x - 25$

Remainder when $\mathrm{p}(x)$ is divided by $x + 3 = \mathrm{p}(-3)$ by the remainder theorem.

$\mathrm{p}(-3) = -27 + 72 + 51 - 25 = 71$

5 **a** Using $x_{n+1} = \sqrt{\dfrac{8{x_n}^2}{3\sec x_n}}$ with $x_1 = 1$:

$x_2 = 1.2003$
$x_3 = 1.1794$
$x_4 = 1.1895$
$x_5 = 1.1849$
$x_6 = 1.1871$
$x_7 = 1.1861$
$x_8 = 1.1865$
$x_9 = 1.1863$
$\alpha = 1.19$ to 2 decimal places

b
$$x = \sqrt{\frac{8x^2}{3\sec x}}$$
$$x^2 = \frac{8x^2}{3\sec x}$$
$$3x^2 \sec x = 8x^2$$
$$x^2(3\sec x - 8) = 0$$
$$x > 0$$
$$\sec x = \frac{8}{3}$$
$$\cos x = \frac{3}{8}$$
$x = 1.1863996$ to 7 decimal places

6 **a**
$$x = \ln(2t + 1)$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{2t + 1}$$
$$y = t - \mathrm{e}^{2t}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - 2\mathrm{e}^{2t}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = \left(1 - 2\mathrm{e}^{2t}\right) \times \frac{2t + 1}{2}$$
$$= \frac{\left(1 - 2\mathrm{e}^{2t}\right)(2t + 1)}{2}$$

b Gradient at $t = 0$:
$$\frac{\left(1 - 2\mathrm{e}^{0}\right)(1)}{2} = -\frac{1}{2}$$
Gradient of normal $= -\dfrac{1}{\left(-\dfrac{1}{2}\right)} = 2$

$$t = 0 \Rightarrow x = \ln 1 = 0$$
$$t = 0 \Rightarrow y = 0 - \mathrm{e}^0 = -1$$
Equation of normal:
$$y - (-1) = 2(x - 0)$$
$$y + 1 = 2x$$
$$y = 2x - 1$$

7 **a** $8\sin\theta + 6\cos\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$$R\cos\alpha = 8$$
$$R\sin\alpha = 6$$
$$R^2 = 6^2 + 8^2 = 100$$
$$R = 10$$
$$\tan\alpha = \frac{6}{8}$$

$\alpha = 36.87^\circ$ to 2 decimal places

$8\sin\theta + 6\cos\theta = 10\sin(\theta + 36.87^\circ)$

b Using the result from part **a**:

$10\sin(\theta + 36.87^\circ) = 7$

$\sin(\theta + 36.87^\circ) \quad = \dfrac{7}{10}$

$\theta + 36.87^\circ$

$= 44.42700...$ or $135.57299...$

$\theta = 7.56^\circ, 98.7^\circ$

c $10\sin(\theta + 36.87^\circ) + 3$

has greatest value

$10 + 3 = 13$

8 **a** $\cos 3x$

$= \cos(2x + x)$

$= \cos 2x\cos x - \sin 2x\sin x$

$= (2\cos^2 - 1)\cos x - 2\sin x\cos x\sin x$

$= 2\cos^3 x - \cos x - 2\sin^2 x\cos x$

$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$

$= 4\cos^3 x - 3\cos x$

b From part **a**:

$4\cos^3 x = \cos 3x + 3\cos x$

$\cos^3 x = \dfrac{1}{4}(\cos 3x + 3\cos x)$

$\displaystyle\int_0^{\frac{\pi}{2}} \cos^3 x \mathrm{d}x$

$\displaystyle= \int_0^{\frac{\pi}{2}} \frac{1}{4}(\cos 3x + 3\cos x)\mathrm{d}x$

$\displaystyle= \left[\frac{1}{12}\sin 3x + \frac{3}{4}\sin x\right]_0^{\frac{\pi}{2}}$

$\displaystyle= \frac{1}{12}\sin\frac{3\pi}{2} + \frac{3}{4}\sin\frac{\pi}{2} - 0 - 0$

$\displaystyle= \frac{1}{12}(-1) + \frac{3}{4}$

$\displaystyle= \frac{3}{4} - \frac{1}{12}$

$\displaystyle= \frac{2}{3}$

PURE MATHEMATICS 3 PRACTICE EXAM-STYLE PAPER

P3 **This exercise is for Pure Mathematics 3 students only.**

All worked solutions within this resource have been written by the author. In examinations, the way marks are awarded may be different.

1

$$\begin{aligned} e^{2x} &= 2^{x+5} \\ \ln e^{2x} &= \ln 2^{x+5} \\ 2x &= (x+5)\ln 2 \\ 2x &= x\ln 2 + 5\ln 2 \end{aligned}$$

$$(2-\ln 2)x = 5\ln 2$$

$$x = \frac{5\ln 2}{2-\ln 2} = 2.652 \text{ to 3 decimal places}$$

2

> When using integration by parts, you usually choose to differentiate powers of x so that they eventually reduce to a constant. However, if there is a $\ln x$ term you need to differentiate it so choose $u = \ln x$

Using integration by parts:

$$\int_0^5 x\ln x\,dx$$

$$= \left[\frac{1}{2}x^2\ln x\right]_0^5 - \int_0^5 \frac{1}{2}x^2 \times \frac{1}{x}dx$$

$$= \frac{25}{2}\ln 5 - 0 - \int_0^5 \frac{1}{2}x\,dx$$

$$= \frac{25}{2}\ln 5 - \left[\frac{1}{4}x^2\right]_0^5$$

$$= \frac{25}{2}\ln 5 - \frac{25}{4} + 0$$

$$= \frac{25}{4}(2\ln 5 - 1)$$

$$= \frac{25}{4}(\ln 5^2 - 1)$$

$$= \frac{25}{4}(\ln 25 - 1)$$

3 $f(x) = 2x^3 - 9x^2 + ax + b$

By the factor theorem:

$$f(4) = 0$$

$$128 - 144 + 4a + b = 0$$

$4a + b = 16$ ....[1]

By the remainder theorem:

$$f(1) = -12$$

$$2 - 9 + a + b = -12$$

$a + b = -5$ ....[2]

[1] − [2]:

$$3a = 21$$

$$a = 7$$

Then from [2]:

$$7 + b = -5$$

$$b = -12$$

4

$$\begin{aligned} 2\sin(x-60^\circ) &= 3\cos x \\ 2\left(\sin x\cos 60^\circ - \cos x\sin 60^\circ\right) &= 3\cos x \\ 2\left(\frac{1}{2}\right)\sin x - 2\left(\frac{\sqrt{3}}{2}\right)\cos x &= 3\cos x \\ \sin x - \sqrt{3}\cos x &= 3\cos x \\ \sin x &= \left(3+\sqrt{3}\right)\cos x \\ \frac{\sin x}{\cos x} &= 3+\sqrt{3} \\ \tan x &= 3+\sqrt{3} \end{aligned}$$

$x = 78.1^\circ$ or $x = 78.1^\circ - 180^\circ = -101.9$

> Here, we have used the fact that the tangent function repeats all values every 180˚. If you have one solution then you can find others by adding or subtracting 180˚.

5

$$\begin{aligned} y^2 &= 4x^2 - x^4 + 5 \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 8x - 4x^3 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{8x-4x^3}{2y} \end{aligned}$$

At P and at Q:

$$\begin{aligned} \frac{8x-4x^3}{2y} &= 0 \\ 8x - 4x^3 &= 0 \\ 4x\left(2-x^2\right) &= 0 \end{aligned}$$

$x = 0$ or $x = \pm\sqrt{2}$

At P, $x = -\sqrt{2}$

$$\begin{aligned} y^2 &= 4\left(-\sqrt{2}\right)^2 - \left(-\sqrt{2}\right)^4 + 5 \\ &= 8 - 4 + 5 = 9 \\ y &= 3 \end{aligned}$$

$P\left(-\sqrt{2}, 3\right)$

$Q\left(\sqrt{2}, -3\right)$, using symmetry about both axes.

$$\begin{aligned} PQ^2 &= \left[\sqrt{2} - \left(-\sqrt{2}\right)\right]^2 + [3-(-3)]^2 \\ &= \left(2\sqrt{2}\right)^2 + (6)^2 \\ &= 8 + 36 = 44 \\ PQ &= \sqrt{44} = 2\sqrt{11} \end{aligned}$$

6 **a** $\mathrm{f}(x) = \dfrac{7x^2-5x+27}{(x-2)\left(x^2+5\right)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+5}$

$7x^2 - 5x + 27 = A\left(x^2+5\right) + (Bx+C)(x-2)$

Letting $x = 2$:

$$\begin{aligned} 28 - 10 + 27 &= 9A + 0 \\ A &= 5 \end{aligned}$$

Letting $x = 0$:

$$\begin{aligned} 27 &= 5A - 2C \\ 27 &= 25 - 2C \\ C &= -1 \end{aligned}$$

Equating coefficients of x^2:

$$\begin{aligned} 7 &= A + B \\ 7 &= 5 + B \\ B &= 2 \end{aligned}$$

$$\mathrm{f}(x) = \frac{7x^2-5x+27}{(x-2)\left(x^2+5\right)} = \frac{5}{x-2} + \frac{2x-1}{x^2+5}$$

b $\dfrac{5}{x-2}+\dfrac{2x-1}{x^2+5}$

$= -5(2-x)^{-1}+(2x-1)\left(5+x^2\right)^{-1}$

$= -\dfrac{5}{2}\left(1-\dfrac{x}{2}\right)^{-1}+\dfrac{1}{5}(2x-1)\left(1+\dfrac{x^2}{5}\right)^{-1}$

$-\dfrac{5}{2}\left(1-\dfrac{x}{2}\right)^{-1}$

$= -\dfrac{5}{2}\left\{1+(-1)\left(-\dfrac{x}{2}\right)+\dfrac{(-1)(-2)}{2!}\left(-\dfrac{x}{2}\right)^2+\ldots\right\}$

$= -\dfrac{5}{2}-\dfrac{5}{4}x-\dfrac{5}{8}x^2+\ldots$

$\dfrac{1}{5}(2x-1)\left(1+\dfrac{x^2}{5}\right)^{-1}$

$= \dfrac{1}{5}(2x-1)\left\{1+(-1)\left(\dfrac{x^2}{5}\right)+\ldots\right\}$

$= \dfrac{1}{5}(2x-1)\left(1-\dfrac{x^2}{5}+\ldots\right)$

$= \dfrac{1}{5}\left\{-1+2x+\dfrac{x^2}{5}-\dfrac{2}{5}x^3+\ldots\right\}$

$-\dfrac{5}{2}\left(1-\dfrac{x}{2}\right)^{-1}+\dfrac{1}{5}(2x-1)\left(1+\dfrac{x^2}{5}\right)^{-1}$

$= \left(-\dfrac{5}{2}-\dfrac{5}{4}x-\dfrac{5}{8}x^2+\ldots\right)+\dfrac{1}{5}\left\{-1+2x+\dfrac{x^2}{5}-\dfrac{2}{5}x^3+\ldots\right\}$

$= \left(-\dfrac{5}{2}-\dfrac{1}{5}\right)+\left(-\dfrac{5}{4}+\dfrac{2}{5}\right)x+\left(-\dfrac{5}{8}+\dfrac{1}{25}\right)x^2+\ldots$

$= -\dfrac{27}{10}-\dfrac{17}{20}x-\dfrac{117}{200}x^2+\ldots$

7 **a** $y = x\sin 2x$

Using the product rule:

$\dfrac{dy}{dx} = \sin 2x + 2x\cos 2x$

$\dfrac{d^2y}{dx^2} = 2\cos 2x + 2\cos 2x - 4x\sin 2x$

$= 4\cos 2x - 4x\sin 2x$

$x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2\left(1+2x^2\right)y$

$= x^2(4\cos 2x-4x\sin 2x)-2x(\sin 2x+2x\cos 2x)+2\left(1+2x^2\right)x\sin 2x$

$= 4x^2\cos 2x-4x^3\sin 2x-2x\sin 2x-4x^2\cos 2x+2x\sin 2x+4x^3\sin 2x$

$= 0$

b Using integration by parts:

$\int_0^{\frac{\pi}{2}} x\sin 2x\,dx$

$= \left[-\dfrac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}-\dfrac{1}{2}\cos 2x\,dx$

$= -\dfrac{\pi}{4}(-1)+0+\left[\dfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$

$= \dfrac{\pi}{4}$

8 **a** $3u - iw = 15$ ....[1]

$u + w = 5 + 10i$

$3u + 3w = 15 + 30i$ ....[2]

[2] − [1]:

$$
\begin{aligned}
3w + \mathrm{i}w &= 30\mathrm{i} \\
w(3+\mathrm{i}) &= 30\mathrm{i} \\
w &= \frac{30\mathrm{i}}{3+\mathrm{i}} \\
&= \frac{30\mathrm{i}(3-\mathrm{i})}{(3+\mathrm{i})(3-\mathrm{i})} \\
&= \frac{90\mathrm{i} - 30\mathrm{i}^2}{9-\mathrm{i}^2} \\
&= \frac{90\mathrm{i}+30}{10} \\
&= 3 + 9\mathrm{i}
\end{aligned}
$$

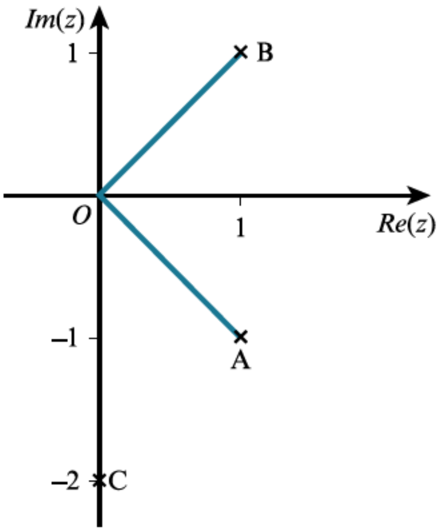
$$
\begin{aligned}
u &= 5 + 10\mathrm{i} - w \\
&= 5 + 10\mathrm{i} - 3 - 9\mathrm{i} \\
&= 2 + \mathrm{i}
\end{aligned}
$$

b $z = \sqrt{2}\left(\cos\frac{-\pi}{4} + \mathrm{i}\sin\frac{-\pi}{4}\right)$

$= 1 - \mathrm{i}$

$z^* = 1 + \mathrm{i}$

$$
\begin{aligned}
z^2 &= (1-\mathrm{i})^2 \\
&= 1 - 2\mathrm{i} + \mathrm{i}^2 \\
&= -2\mathrm{i}
\end{aligned}
$$



Longest side $= BC$

$BC^2 = 1^2 + 3^2 = 10$

$BC = \sqrt{10}$

9 **a** $(5\mathbf{i} + 7\mathbf{j} + p\mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} + p\mathbf{k}) = 0$

$$
\begin{aligned}
5 - 14 + p^2 &= 0 \\
p^2 &= 9 \\
p &= \pm 3
\end{aligned}
$$

b **i** $\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -6 \\ 1 \end{pmatrix}$

ii If the lines intersect:

$$\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 9 \\ 1 \end{pmatrix}$$

$5 + s = 3t$

$s - 3t = -5$ ....[1]

$-6s = 4 + 9t$

$6s + 9t = -4$ ....[2]

$2 + s = 3 + t$

$s - t = 1$ .... [3]

[3] – [1]:

$2t = 6$

$t = 3$

Then from [3]:

$s - 3 = 1$

$s = 4$

Checking in [2]:

$$\begin{aligned} &6s + 9t \\ &= 24 + 27 \neq -4 \end{aligned}$$

Then equations are inconsistent, so the lines do not intersect.

10 **a**

$$\begin{aligned} \frac{dx}{dt} &= \alpha(100 - x) \\ \frac{1}{100 - x}\frac{dx}{dt} &= \alpha \\ \int \frac{1}{100 - x}\frac{dx}{dt}dx &= \int \alpha dt \\ -\ln|100 - x| &= \alpha t + c \end{aligned}$$

Using the fact that $x = 25$ when $t = 0$:

$$\begin{aligned} -\ln 75 &= 0 + c \\ c &= -\ln 75 \\ -\ln|100 - x| &= \alpha t - \ln 75 \\ \ln|100 - x| - \ln 75 &= -\alpha t \\ \ln\left|\frac{100 - x}{75}\right| &= -\alpha t \end{aligned}$$

Assuming $x < 100$:

$$\begin{aligned} -\ln\left(\frac{100 - x}{75}\right) &= -\alpha t \\ \ln\left(\frac{75}{100 - x}\right) &= -\alpha t \end{aligned}$$

b

$$\begin{aligned} \frac{100 - x}{75} &= e^{-\alpha t} \\ 100 - x &= 75e^{-\alpha t} \\ x &= 100 - 75e^{-\alpha t} \end{aligned}$$

Using the fact that $x = 500\alpha$ when $t = 2$:

$$\begin{aligned} 500\alpha &= 100 - 75e^{-2\alpha} \\ \alpha &= 0.2 - 0.15e^{-2\alpha} \end{aligned}$$

c Using $\alpha_{n+1} = 0.2 - 0.15e^{-2\alpha_n}$ with $\alpha_0 = 0.1$:

$\alpha_1 = 0.07719$

$\alpha_2 = 0.07146$

$\alpha_3 = 0.06998$

$\alpha_4 = 0.06959$

$\alpha_5 = 0.06949$

$\alpha_6 = 0.06946$

$\alpha = 0.069$ to 2 significant figures

d $x = 100 - 75e^{-0.069t}$

When $t = 30$:

$$\begin{aligned} x &= 100 - 75e^{-0.069(30)} \\ &= 90.5^\circ C \end{aligned}$$

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